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Two-level method and some a priori estimates in unsteady Navier–Stokes calculations

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Abstract

A two-level method proposed for quasielliptic problems is adapted in this paper to the simulation of unsteady incompressible Navier–Stokes flows. The method requires a solution of a nonlinear problem on a coarse grid and a solution of linear symmetric problem on a fine grid, the scaling between these two grids is superlinear. Approximation, stability, and convergence aspects of a fully discrete scheme are considered. Stability properties of the two-level scheme are compared with those for a commonly used semi-implicit scheme, some new estimates are also proved for the latter. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Numerical simulation of unsteady incompressible viscous flow is a fundamental problem both of numerical analyses and fluid dynamics. The governing equations are the incompressible Navier–Stokes ones:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = 0, \end{aligned}\right\} \quad \text{in } \Omega\times(0,T], \tag{1.1}$$

with a given force field $\boldsymbol{f}$, kinematic viscosity $\nu > 0$, $\Omega \subset \mathbb{R}^n$, $n = 2,3$. The velocity vector function $\boldsymbol{u}$ and pressure scalar function p to be found are subject to some boundary conditions, which we

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assume to be Dirichlet and homogeneous for the velocity:

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega \times [0, T] \tag{1.2}$$

and initial condition at $t = 0$:

$$\boldsymbol{u} = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \bar{\Omega}. \tag{1.3}$$

Another common assumption is $\int_\Omega p(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = 0$ for the pressure $p(\boldsymbol{x}, t)$ for all $t \in (0, T]$. For a detailed consideration of mechanical, mathematical, and computational aspects associated with Navier–Stokes problem we refer to, among others, [9,12,13,25].

Belonging of a weak solution of (1.1)–(1.3) to a space of solenoidal functions and domination of nonlinear convection terms in the momentum equation for moderate and high Reynolds numbers (small ν and/or high velocities) are commonly considered to be the main difficulties in numerical and computational theory of Navier–Stokes equations. The objective of this paper is to apply two-level mesh reduction method to the treatment of convection phenomenon in unsteady numerical simulations. The method is closely related to the nonlinear Galerkin method [1,18–20] and was developed in [28,3,14–16].

Using numerous solution schemes for (1.1)–(1.3) with $\boldsymbol{u}-p$ coupling (see, e.g., [5,11]) or operator splitting [6], one has to choose between a fully implicit treatment of nonlinear terms in (1.1) or some of their linearization using, e.g. extrapolation in time. In the first case one faces the necessity of solving nonlinear problem of the Burgers or Navier–Stokes type on each time step, otherwise linear and even symmetric problems can be obtained on each time step. However, the latter approach may cause stability problems, i.e. time step τ becomes subject to some conditions involving spatial discretization and/or Reynolds number. We refer to [25] for theoretical considerations and [27] for experimental comparison of various schemes.

The method presented and studied here can be roughly described as follows. For numerical solution of (1.1)–(1.3) choose some spatial finite element or finite difference discretization and two meshes: the coarse one with the step H and the fine one with the step h such that $h \sim H^\alpha$, $\alpha \geqslant 1$. For temporal integration on $[t_0, t_0 + \tau]$ with time step $\tau > 0$ make fully implicit step on the coarse grid and obtain $\boldsymbol{u}_H, p_H$ for $t_0 + \tau$ via solution of nonlinear problem on the coarse grid. Then extrapolate $\boldsymbol{u}_H$ on the fine grid. Using linearization of convective terms about $\boldsymbol{u}_H$, make semi-implicit integration step on the fine grid and obtain $\boldsymbol{u}_h$, p_h for $t_0 + \tau$ solving linear symmetric problem on the fine grid.

Compared to the recent studies [1,20] for unsteady problems, the primary innovations in this paper are the treatment of fully discrete (both in time and space) case of the method. Hence the stability results are quite important. We also avoid the use of any intermediate finite element subspaces. Therefore well established solvers can be readily applied to the auxiliary finite element problems.

In Section 2 of the paper we introduce necessary notations and preliminary results. The algorithm to be studied is described in Section 3. Where possible we compare results obtained for the constructed algorithm with appropriate ones for commonly used algorithm based on a fully explicit treatment of the nonlinearity via extrapolation in time. For this purpose some results on stability of the latest algorithm are also proved.

In Section 4 some approximation results are proved. In particular, it follows that in the case of linear velocity – constant pressure finite elements the scaling $h \sim H^2$ gives the same order of spatial discretization error as the usual Galerkin method with mesh size h. We note that this relation between coarse and fine grids is somewhat less impressive than the one recovered in the framework of Newton-type methods (see [2,28], and references cited therein). However, application of the latter techniques for the problem considered requires highly nonsymmetrical problems to be solved on each time step. Moreover, it causes the appearance of undesirable reactive term in the linearized equation (see also [16]).

The stability of the schemes is studied in Section 5. It is proved that while usual semi-implicit scheme requires *time* step to be small enough to guarantee the stability, the two-level scheme (at least theoretically) requires the *spatial* step to be small enough to ensure stability. Moreover, if the problem is regular enough, the use of high order finite elements weakens the condition on h. In Section 6 we consider convergence of the two-level scheme. The appropriate convergence for velocity is proved in two dimensions.

Throughout the paper we deal with saddle point formulations of the corresponding finite element problems (discrete velocity is not solenoidal in general). This causes some extra complications but corresponds to real-life situations.

2. Preliminaries

Throughout the paper we assume Ω to be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary, or a convex polygon (polyhedron).

Later on we need the following functional spaces:

$$\boldsymbol{H}_0^1 \equiv \{\boldsymbol{u} \in W_2^1(\Omega)^2\colon \boldsymbol{u} = 0 \text{ on } \partial\Omega\},$$
$$\boldsymbol{V} \equiv \{\boldsymbol{u} \in \boldsymbol{H}_0^1\colon \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}$$

with energy scalar product $(\boldsymbol{u},\boldsymbol{v})_1 = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}),\ \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1$,

$$\boldsymbol{L}^0 \equiv \{\boldsymbol{u} \in L_2(\Omega)^2\colon \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$L_2/R \equiv \left\{p \in L_2(\Omega)\colon \int_\Omega p\,\mathrm{d}x = 0\right\}$$

with L_2-scalar product. Let $\boldsymbol{H}^{-1}$ be a dual, with respect to L_2-duality, space to $\boldsymbol{H}_0^1$ with the corresponding norm:

$$\|\boldsymbol{f}\|_{-1} = \sup_{0\neq \boldsymbol{u}\in \boldsymbol{H}_0^1} \frac{\langle \boldsymbol{f},\boldsymbol{u}\rangle}{\|\boldsymbol{u}\|_1}, \quad \boldsymbol{f} \in \boldsymbol{H}^{-1}.$$

We also use Sobolev spaces of a real exponent s: $H^s(\Omega)$ with a norm $\|\cdot\|_s$.

The following forms are associated with the Navier–Stokes problem:

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v})_1, \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1,$$
$$a_\tau(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v}) + \nu\tau(\boldsymbol{u},\boldsymbol{v})_1, \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1,\ \tau > 0,$$
$$b(p,\boldsymbol{u}) = (p, \operatorname{div} \boldsymbol{u}), \quad p \in L_2/R,\ \boldsymbol{u} \in \boldsymbol{H}_0^1,$$
$$N(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \tfrac{1}{2}[((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v})], \quad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1.$$

The weak formulation of (1.1)–(1.3) is to find

$$\boldsymbol{u} \in L_2(0,T;\boldsymbol{H}_0^1) \cap L_\infty(0,T;\boldsymbol{L}^0) \quad \text{and} \quad p \in L_2(0,T;L_2/R)$$

satisfying for given $\boldsymbol{f} \in L_2(0,T;\boldsymbol{H}^{-1})$ in the sense of distributions on $(0,T]$

$$\left(\frac{\partial \boldsymbol{u}}{\partial t},\boldsymbol{v}\right) + \nu\, a(\boldsymbol{u},\boldsymbol{v}) + N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - b(p,\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle,$$
$$b(q,\boldsymbol{u}) = 0 \quad \forall\{\boldsymbol{v},q\} \in \boldsymbol{H}_0^1 \times L_2/R,$$
$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \bar{\Omega},$$

or, alternatively, to find

$$\boldsymbol{u} \in L_2(0,T;\boldsymbol{V}) \cap L_\infty(0,T;\boldsymbol{L}^0)$$

satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u},\boldsymbol{v}) + \nu\, a(\boldsymbol{u},\boldsymbol{v}) + N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$
$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \bar{\Omega},$$

De Rham theorem connects both formulations (cf. [9,25]).

It is worth mentioning that a weak solution defined above exists and for $n=2$ is unique; moreover, some extra assumptions on $\boldsymbol{f}$, $\partial \boldsymbol{f}/\partial t$, and $\boldsymbol{u}_0$ provide $\boldsymbol{u} \in L_2(0,T;H^2(\Omega)^2)$ (cf. [17]).

Further, we shall use the following estimates due to [12,25]:

$$\begin{aligned}
&b(p,\boldsymbol{u}) \leqslant \|p\|_0 \|\boldsymbol{u}\|_1 && \forall p \in L_2/R,\ \boldsymbol{u} \in \boldsymbol{H}_0^1,\\
&(\boldsymbol{f},\boldsymbol{u}) \leqslant \|\boldsymbol{f}\|_{-1} \|\boldsymbol{u}\|_1 && \forall \boldsymbol{f} \in \boldsymbol{H}^{-1},\ \boldsymbol{u} \in \boldsymbol{H}_0^1,\\
&|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})| \leqslant c\, \|\boldsymbol{u}\|_{L_4} \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_{L_4} && \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1,\\
&|N(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c\, \|\boldsymbol{u}\|_s \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_1 && \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1,\\
&|N(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c\, \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_s \|\boldsymbol{w}\|_1 && \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1,\\
&|N(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c\, \|\boldsymbol{u}\|_0 \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_{1+s} && \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1,\ \boldsymbol{w} \in \boldsymbol{H}_0^1 \cap H^{1+s}.
\end{aligned} \tag{2.1}$$

From now on we agree to consider s as an arbitrary number from $(0,1]$ for two-dimensional problem and $s \in [\frac{1}{2},1]$ for three-dimensional problem, if it is not stated otherwise. We also need

$$\begin{aligned}
&\|\boldsymbol{u}\|_{L_4}^2 \leqslant \sqrt{2}\, \|\boldsymbol{u}\|_0 \|\boldsymbol{u}\|_1 && \forall \boldsymbol{u} \in \boldsymbol{H}_0^1,\ \Omega \in R^2,\\
&\|\boldsymbol{u}\|_{L_4}^2 \leqslant 2\, \|\boldsymbol{u}\|_0^{1/2} \|\boldsymbol{u}\|_1^{3/2} && \forall \boldsymbol{u} \in \boldsymbol{H}_0^1,\ \Omega \in R^3,\\
&\|\boldsymbol{u}\|_s \leqslant c\, \|\boldsymbol{u}\|_0^{1-s} \|\boldsymbol{u}\|_1^s && \forall \boldsymbol{u} \in \boldsymbol{H}_0^1,\ s \in (0,1).
\end{aligned} \tag{2.2}$$

Here and later on we denote by $c(\Omega), c, c_0, c_1, \ldots$ some constants independent of both spatial and temporal discretization parameters (h and τ) and ν, otherwise we shall use, for example, $c(\nu)$ for a constant depending possibly on ν.

So called ϵ-inequality:

$$|ab| \leqslant \frac{1}{2\epsilon}|a|^2 + \frac{\epsilon}{2}|b|^2, \quad \forall a,b \in R,\ \epsilon > 0,$$

will be also used throughout the paper.

Let us denote by h a mesh size parameter, and denote by $\boldsymbol{H}_h$ a finite element subspace of $\boldsymbol{H}_0^1$ and by Q_h a finite element subspace of L_2/R. Assume that for some real $h_0 > 0$, positive integers k_1, k_2, and $h \in (0, h_0]$ the following hypotheses hold. Examples of such finite element spaces can be found in [5,9,11].

(H1) Approximation hypothesis

(a) For all $\boldsymbol{u} \in \boldsymbol{H}_0^1 \cap H^d(\Omega)^n$, with integer $d > 0$

$$\inf_{\boldsymbol{v}\in \boldsymbol{H}_h} \|\boldsymbol{u}-\boldsymbol{v}\|_1 \leqslant ch^l \|\boldsymbol{u}\|_d, \quad l = \min(k_1, d-1).$$

(b) For all $p \in L_2/R \cap H^d(\Omega)$, with integer $d > 0$

$$\inf_{q\in Q_h} \| p-q \|_0 \leqslant ch^l \| p \|_d, \quad l = \min(k_2, d).$$

(H2) Inverse hypothesis

For all $\boldsymbol{u} \in \boldsymbol{H}_h$

$$\|\boldsymbol{u}\|_1 \leqslant \kappa h^{-1} \|\boldsymbol{u}\|_0 .$$

(H3) Stability hypothesis. There exists some real constant $c_0 > 0$ independent on h such that

$$\sup_{0\neq \boldsymbol{u}\in \boldsymbol{H}_h} \frac{b(p,\boldsymbol{u})}{\|\boldsymbol{u}\|_1} \geqslant c_0 \| p \|_0, \quad \forall p \in Q_h.$$

The above hypotheses give the following standard result concerning Stokes problem.

Lemma 2.1. *For any given $\{\boldsymbol{u}, p\} \in \boldsymbol{H}_0^1 \times L_2/R$, there exist unique $\boldsymbol{u}_h \in \boldsymbol{H}_h$ and $p_h \in Q_h$ such that*

$$a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}) - b(p-p_h, \boldsymbol{v}) = 0,$$

$$b(q, \boldsymbol{u}-\boldsymbol{u}_h) = 0, \quad \forall \{\boldsymbol{v}, q\} \in \boldsymbol{H}_h \times Q_h.$$

Moreover,

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \| p-p_h \|_0 \leqslant c(\Omega)\left(\inf_{\boldsymbol{v}\in \boldsymbol{H}_h} \|\boldsymbol{u}-\boldsymbol{v}\|_1 + \inf_{q\in Q_h} \| p-q \|_0 \right),$$

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \leqslant c(\Omega) h \inf_{\boldsymbol{v}\in \boldsymbol{H}_h} \|\boldsymbol{u}-\boldsymbol{v}\|_1 .$$

Denote also by $\boldsymbol{V}_h$ a subspace of discretely divergence-free functions from $\boldsymbol{H}_h$: $\boldsymbol{V}_h = \{\boldsymbol{u}_h \in \boldsymbol{H}_h\colon b(q, \boldsymbol{u}_h) = 0 \ \forall q \in Q_h\}$.

3. Two algorithms for the unsteady problem

The common and effective way of numerical treatment of time-dependent problems is separation of spatial and temporal discretizations. For spatial discretization one can choose finite element, finite difference, spectral methods, while for the temporal discretization the finite difference method is the most natural choice. Two schemes described below utilize this idea.

First let us consider one widely used semi-implicit scheme for solving unsteady Navier–Stokes problem (1.1)–(1.3). For given $\boldsymbol{u}_h^0 \in \boldsymbol{H}_h$ find $\{\boldsymbol{u}_h^{i+1}, p_h^{i+1}\} \in \boldsymbol{H}_h \times Q_h$ for $i = 0, 1, \ldots$ from the relations

$$a_{\tau_i}(\boldsymbol{u}_h^{i+1},\boldsymbol{v})-\tau_i b(p_h^{i+1},\boldsymbol{v})+\tau_i N(\boldsymbol{u}_h^i;\boldsymbol{u}_h^i,\boldsymbol{v})=\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{v})+(\boldsymbol{u}_h^i,\boldsymbol{v}),$$
$$b(q,\boldsymbol{u}_h^{i+1})=0\quad\forall\{\boldsymbol{v},q\}\in\boldsymbol{H}_H\times Q_H. \tag{3.1}$$

where $\tau_i>0$ are steps of temporal discretization, which we assume generally to be variable. In (3.1) and below we set $\boldsymbol{f}^{i+1}=\tau_i^{-1}\int_{t_1}^{t_2}\boldsymbol{f}\,\mathrm{d}t$ with $t_2=\sum_{k=0}^{i}\tau_k,\ t_1=t_2-\tau_i$.

On every time step scheme (3.1) requires solution of the generalized Stokes problem, effective solution methods for such problem are available [22,8]. With the above assumptions scheme (3.1) is conditionally stable and the sufficient conditions for stability from [25] are $\tau\leqslant c(\nu,\Omega,\boldsymbol{f}(t),\boldsymbol{u}^0)h^n$, ($\Omega\subset R^n$); moreover, the careful reading of the proof gives $c=\mathrm{O}(\nu^2),\ \nu\to 0$.

The scheme being proposed works as follows. For given $\boldsymbol{u}_h^0\in\boldsymbol{H}_h$ find $\{\boldsymbol{u}_h^{i+1},p_h^{i+1}\}\in\boldsymbol{H}_h\times Q_h$ for $i=0,1,\dots$ from the two-level algorithm:

Find $\{\boldsymbol{u}_H^{i+1},p_H^{i+1}\}\in\boldsymbol{H}_H\times Q_H$

$$a_{\tau_i}(\boldsymbol{u}_H^{i+1},\boldsymbol{v})-\tau_i b(p_H^{i+1},\boldsymbol{v})+\tau_i N(\boldsymbol{u}_H^{i+1};\boldsymbol{u}_H^{i+1},\boldsymbol{v})=\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{v})+(\boldsymbol{u}_h^i,\boldsymbol{v}),$$
$$b(\boldsymbol{u}_H^{i+1},q)=0\quad\forall\{\boldsymbol{v},q\}\in\boldsymbol{H}_H\times Q_H; \tag{3.2a}$$

with known $\{\boldsymbol{u}_H^{i+1},p_H^{i+1}\}$ find $\{\boldsymbol{u}_h^{i+1},p_h^{i+1}\}\in\boldsymbol{H}_h\times Q_h$

$$a_{\tau_i}(\boldsymbol{u}_h^{i+1},\boldsymbol{v})-\tau_i b(p_h^{i+1},\boldsymbol{v})+\tau_i N(\boldsymbol{u}_H^{i+1};\boldsymbol{u}_H^{i+1},\boldsymbol{v})=\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{v})+(\boldsymbol{u}_h^i,\boldsymbol{v}),$$
$$b(\boldsymbol{u}_h^{i+1},q)=0\quad\forall\{\boldsymbol{v},q\}\in\boldsymbol{H}_h\times Q_h. \tag{3.2b}$$

The above two-level algorithm can be observed simultaneously from two points of view. On the one hand it can be considered as an improvement of scheme (3.1) by obtaining a qualitative information (on a coarse grid) about the solution at time $(t_0+\tau_i)$. On the other hand, let us consider fully implicit unconditionally stable scheme for (1.1)–(1.3) without any spatial discretization: for given $\boldsymbol{u}^0\in\boldsymbol{H}_0^1$ find $\{\boldsymbol{u}^{i+1},p^{i+1}\}\in\boldsymbol{H}_0^1\times L_2/R$ for $i=0,1,\dots$ from

$$a_{\tau_i}(\boldsymbol{u}^{i+1},\boldsymbol{v})-\tau_i b(p^{i+1},\boldsymbol{v})+\tau_i N(\boldsymbol{u}^{i+1};\boldsymbol{u}^{i+1},\boldsymbol{v})=\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{v})+(\boldsymbol{u}^i,\boldsymbol{v}),$$
$$b(\boldsymbol{u}^{i+1},q)=0\quad\forall\{\boldsymbol{v},q\}\in\boldsymbol{H}_0^1\times L_2/R \tag{3.3}$$

then (3.2a) and (3.2b) is a straightforward application of the two-level method from [15] to the solution of steady nonlinear problem arising on every time step of (3.3). Note that the two-level algorithm (3.2) requires on every time step solution of the nonlinear problem of Navier–Stokes type on the coarse grid and solution of the linear symmetric problem of Stokes type on the fine grid.

Remark 3.1. To improve the accuracy and stability of scheme (3.1) such variants of (3.1) as Crank–Nicolson, fractional-step [21], with high-order extrapolation in time of nonlinear terms, multistep [4], with upwinding (see, e.g. [24]) are known and used in practice. We note that all or at least most of these improvements are quite applicable to (3.2a) and (3.2b) as well as splitting techniques leading to a class of projection type methods [6,10].

4. Approximation

Consider the following problem. For given $\boldsymbol{g} \in \boldsymbol{H}^{-1}$ find $\{\boldsymbol{u}, p\} \in \boldsymbol{H}_0^1 \times L_2/R$ satisfying

$$\begin{aligned} &a_\tau(\boldsymbol{u},\boldsymbol{v}) - \tau b(p,\boldsymbol{v}) + \tau N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{g},\boldsymbol{v}),\\ &b(q,\boldsymbol{u}) = 0 \quad \forall\{\boldsymbol{v},q\} \in \boldsymbol{H}_0^1 \times L_2/R. \end{aligned} \tag{4.1}$$

Solution of problem (4.1) exists (see, e.g. [25, Lemma IV.4.3]). For the uniqueness it is sufficient $\sqrt{2} \parallel g \parallel_{-1} \leqslant \tau^{1/2}\nu^{3/2}$ for two-dimensional problem and $2\sqrt{2} \parallel g \parallel_{-1} \leqslant \tau^{3/4}\nu^{7/4}$ for three-dimensional problem. If $g \in L^n(\Omega)$ then sufficient conditions can be written as $\parallel g \parallel_0 \leqslant \nu$ and $2 \parallel g \parallel_0 \leqslant \tau^{1/2}\nu^{3/2}$, respectively. The proof is quite standard, it follows from (2.1), (2.2) and a priori estimates for weak solutions of (4.1):

$$2 \parallel u \parallel_0^2 + \tau\nu \parallel u \parallel_1^2 \leqslant (\tau\nu)^{-1} \parallel g \parallel_{-1}^2,$$

$$\parallel u \parallel_0^2 + 2\tau\nu \parallel u \parallel_1^2 \leqslant \parallel g \parallel_0^2$$

and certain relation for the difference $\boldsymbol{w} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$ between two weak solutions of (4.1) that can be obtained from (4.1) with $\boldsymbol{v} = \boldsymbol{w}$.

Two-level method for problem (4.1) means: find sequentially $\{\boldsymbol{u}_H, p_H\} \in \boldsymbol{H}_H \times Q_H$ and $\{\boldsymbol{u}_h, p_h\} \in \boldsymbol{H}_h \times Q_h$ from

$$\begin{aligned} &a_\tau(\boldsymbol{u}_H,\boldsymbol{v}) - \tau b(p_H,\boldsymbol{v}) + \tau N(\boldsymbol{u}_H;\boldsymbol{u}_H,\boldsymbol{v}) = (\boldsymbol{g},\boldsymbol{v}),\\ &b(q,\boldsymbol{u}_H) = 0 \quad \forall\{\boldsymbol{v},q\} \in \boldsymbol{H}_H \times Q_H \end{aligned} \tag{4.2a}$$

and

$$\begin{aligned} &a_\tau(\boldsymbol{u}_h,\boldsymbol{v}) - \tau b(p_h,\boldsymbol{v}) + \tau N(\boldsymbol{u}_H;\boldsymbol{u}_H,\boldsymbol{v}) = (\boldsymbol{g},\boldsymbol{v}),\\ &b(q,\boldsymbol{u}_h) = 0 \quad \forall\{\boldsymbol{v},q\} \in \boldsymbol{H}_h \times Q_h. \end{aligned} \tag{4.2b}$$

Theorem 4.1. *For given* $\boldsymbol{g} \in \boldsymbol{H}^{-1}$ *let* $\{\boldsymbol{u}, p\} \in \boldsymbol{H}_0^1 \cap H^{1+s}(\Omega)^n \times L_2/R$, $n = 2,3$ *be a solution to problem* (4.1) *and let* $\{\boldsymbol{u}_h, p_h\} \in \boldsymbol{H}_h \times Q_h$ *be a solution to problem* (4.2a), (4.2b); *then the following estimate is valid*:

$$\begin{aligned} \parallel \boldsymbol{u}_h - \boldsymbol{u} \parallel_0^2 + \tau\nu \parallel \boldsymbol{u}_h - \boldsymbol{u} \parallel_1^2 + \tau^2 \parallel p_h - p \parallel_0^2 \leqslant C(\Omega,s)\Big((h^2+\tau\nu) \inf_{\boldsymbol{v}\in \boldsymbol{H}_h} \parallel \boldsymbol{u} - \boldsymbol{v} \parallel_1^2 \\ + \gamma_h(\tau,h,\nu)\Big(\inf_{q\in Q_h} \parallel p - q \parallel_0^2 + \parallel \boldsymbol{u}_H - \boldsymbol{u} \parallel_0^2 \parallel \boldsymbol{u} \parallel_{1+s}^2 + \parallel \boldsymbol{u}_H - \boldsymbol{u} \parallel_0^{2-2s} \parallel \boldsymbol{u}_H - \boldsymbol{u} \parallel_1^{2+2s}\Big)\Big), \end{aligned} \tag{4.3}$$

with $\gamma_h = \min(\tau/\nu, \tau^2/h^2)$.

Proof. For given $\{\boldsymbol{u}, p\}$ from (4.1) let $\{\boldsymbol{u}_S, p_S\} \in \boldsymbol{V}_h \times Q_h$ be a solution to the Stokes problem

$$\begin{aligned} &(\nabla \boldsymbol{u}_S, \nabla \boldsymbol{v}) - b(p_S,\boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - b(p,\boldsymbol{v}),\\ &b(\boldsymbol{u}_S,q) = 0, \quad \forall\{\boldsymbol{v},q\} \in \boldsymbol{H}_h \times Q_h, \end{aligned}$$

then, by virtue of Lemma 2.1, the following estimates are valid:

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_S\|_1 &\leqslant c(\Omega)\Big(\inf_{\boldsymbol{v}\in \boldsymbol{H}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_1+\inf_{q\in Q_h}\|p-q\|_0\Big),\\ \|\boldsymbol{u}-\boldsymbol{u}_S\|_0 &\leqslant c(\Omega)h\inf_{\boldsymbol{v}\in \boldsymbol{H}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_1 .\end{aligned} \tag{4.4}$$

We also choose $p_I=\arg\min_{q\in Q_h}\|p-q\|_0$.

We choose in (4.1) and (4.2b) $\boldsymbol{v}=\boldsymbol{u}_S-\boldsymbol{u}_h$ and substract (4.2b) from (4.1). Then we obtain

$$\begin{aligned}\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_0^2+\tau\nu\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1^2 &= a_\tau(\boldsymbol{u}_S-\boldsymbol{u},\boldsymbol{u}_S-\boldsymbol{u}_h)\\ &\quad+\tau(b(p-p_I,\boldsymbol{u}_S-\boldsymbol{u}_h)-N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{u}_S-\boldsymbol{u}_h)+N(\boldsymbol{u}_H;\boldsymbol{u}_H,\boldsymbol{u}_S-\boldsymbol{u}_h))\\ &= a_\tau(\boldsymbol{u}_S-\boldsymbol{u},\boldsymbol{u}_S-\boldsymbol{u}_h)+\tau(b(p-p_I,\boldsymbol{u}_S-\boldsymbol{u}_h)-N(\boldsymbol{u}-\boldsymbol{u}_H;\boldsymbol{u},\boldsymbol{u}_S-\boldsymbol{u}_h)\\ &\quad+N(\boldsymbol{u}-\boldsymbol{u}_H;\boldsymbol{u}-\boldsymbol{u}_H,\boldsymbol{u}_S-\boldsymbol{u}_h)-N(\boldsymbol{u};\boldsymbol{u}-\boldsymbol{u}_H,\boldsymbol{u}_S-\boldsymbol{u}_h))\\ &\leqslant \tfrac12\|\boldsymbol{u}-\boldsymbol{u}_S\|_0^2+\tfrac12\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_0^2+\frac12\tau\nu\|\boldsymbol{u}-\boldsymbol{u}_S\|_1^2+\tfrac12\tau\nu\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1^2\\ &\quad+c\tau(\|p-p_I\|_0+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0\|\boldsymbol{u}\|_{1+s}+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{1+s})\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1 .\end{aligned} \tag{4.5}$$

Using Cauchy inequality, we get

$$\begin{aligned}&c\tau(\|p-p_I\|_0+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0\|\boldsymbol{u}\|_{1+s}+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{1+s})\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1\\ &\quad\leqslant c^2\tau\nu^{-1}(\|p-p_I\|_0^2+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^2\|\boldsymbol{u}\|_{1+s}^2+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{2-2s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{2+2s})\\ &\qquad+\frac{\tau\nu}{4}\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1^2,\end{aligned} \tag{4.6a}$$

and

$$\begin{aligned}&c\tau(\|p-p_I\|_0+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0\|\boldsymbol{u}\|_{1+s}+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{1+s})\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_1\\ &\quad\leqslant \frac{c\tau\kappa}{h}(\|p-p_I\|+\|\boldsymbol{u}-\boldsymbol{u}_H\|\|\boldsymbol{u}\|_{1+s}+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{1+s})\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_0\\ &\quad\leqslant \frac{c^2\tau^2\kappa^2}{h^2}(\|p-p_I\|_0^2+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^2\|\boldsymbol{u}\|_{1+s}^2+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{2-2s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{2+2s})\\ &\qquad+\tfrac14\|\boldsymbol{u}_S-\boldsymbol{u}_h\|_0^2 .\end{aligned} \tag{4.6b}$$

Now in order to estimate $\|\boldsymbol{u}_h-\boldsymbol{u}\|$, we apply the triangle inequality: $\|\boldsymbol{u}_h-\boldsymbol{u}\|\leqslant\|\boldsymbol{u}_h-\boldsymbol{u}_S\|+\|\boldsymbol{u}_S-\boldsymbol{u}\|$, estimates (4.4), (4.5), and one of (4.6a) and (4.6b). To estimate $\|p_h-p\|_0$ we use the following standard arguments (see, e.g. [9]). From (4.1) and (4.2b) we get for all $\{\boldsymbol{v},q\}\in\boldsymbol{H}_h\times Q_h$

$$\tau b(p_h-q,\boldsymbol{v})=a_\tau(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})+\tau b(p-q,\boldsymbol{v})+\tau N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})-\tau N(\boldsymbol{u}_H;\boldsymbol{u}_H,\boldsymbol{v}).$$

Thus, similar to (4.5) we obtain for all $\{\boldsymbol{v},q\}\in\boldsymbol{H}_h\times Q_h$:

$$\begin{aligned}\tau b(p_h-q,\boldsymbol{v})&\leqslant(c+\tau\nu)^{1/2}(\|\boldsymbol{u}_h-\boldsymbol{u}\|_0^2+\tau\nu\|\boldsymbol{u}_h-\boldsymbol{u}\|_1^2)^{1/2}\|\boldsymbol{v}\|_1\\ &\quad+c\tau(\|p-q\|_0+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0\|\boldsymbol{u}\|_{1+s}+\|\boldsymbol{u}-\boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u}-\boldsymbol{u}_H\|_1^{1+s})\|\boldsymbol{v}\|_1 .\end{aligned}$$

Dividing both sides of the last inequality by $\|\boldsymbol{v}\|_1$ for $0 \neq \boldsymbol{v} \in \boldsymbol{H}_h$, taking sup over all such $\boldsymbol{v}$ and inf over all $q \in Q_h$, and utilizing hypothesis (H3) we get

$$c_0\tau \inf_{q\in Q_h} \| p_h - q \|_0 \leqslant (c+\tau\nu)^{1/2}(\|\boldsymbol{u}_h - \boldsymbol{u}\|_0^2 + \tau\nu\|\boldsymbol{u}_h - \boldsymbol{u}\|_1^2)^{1/2}$$
$$+ c\tau\left(\inf_{q\in Q_h} \| p - q\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_H\|_0 \|\boldsymbol{u}\|_{1+s} + \|\boldsymbol{u} - \boldsymbol{u}_H\|_0^{1-s}\|\boldsymbol{u} - \boldsymbol{u}_H\|_1^{1+s}\right).$$

Now from the triangle inequality: $\| p_h - p \|_0 \leqslant \inf_{q\in Q_h} \| p_h - q\|_0 + \inf_{q\in Q_h}\| p - q\|_0$ we get an estimate on $\| p_h - p\|_0$. Combining estimates for $\|\boldsymbol{u}_h - \boldsymbol{u}\|_0^2 + \tau\nu\|\boldsymbol{u}_h - \boldsymbol{u}\|_1^2$ and $\tau^2 \| p_h - p\|_0^2$, we prove the theorem.

Remark 4.2. Estimate (4.3) shows a loss of convergence in pressure for too small time steps τ. Indeed, the same standard arguments as in the proof of Theorem 4.1 when applied to some finite element approximation of linear (generalized) Stokes problem

$$\begin{aligned} &\tau^{-1}\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = 0, \\ &\boldsymbol{u}|_{\partial\Omega} = 0 \end{aligned} \tag{4.7}$$

give the estimate

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_0^2 + \tau\nu\|\boldsymbol{u}_h - \boldsymbol{u}\|_1^2 + \tau^2\| p_h - p\|_0^2$$
$$\leqslant C(\Omega)\left((h^2+\tau\nu)\inf_{\boldsymbol{v}\in \boldsymbol{H}_h}\|\boldsymbol{u} - \boldsymbol{v}\|_1^2 + \gamma_h \inf_{q\in Q_h}\| p - q\|_0^2\right). \tag{4.8}$$

Assumed to be optimal for pressure this estimate recovers the necessity of $h^{2k_2} \ll \tau\nu$ or $h^{2(k_2-1)} \ll 1$ for pressure finite elements and $h^{k_1+1} \ll \tau$ for velocity finite elements to ensure the convergence of pressure in $L_2(\Omega)$. These conditions on h, τ, and ν force us to use schemes of equal order ($k_2 = k_1 + 1 \geqslant 2$) or high order spacial interpolation ($k_2 \geqslant 2, k_1 \geqslant 2$), and implicit schemes, which do not require τ to be too small.

Remark 4.3. Further we shall need an estimate for $\|\boldsymbol{u} - \boldsymbol{u}_H\|_0$, where $\boldsymbol{u}$ is a solution to (4.1) and $\boldsymbol{u}_H$ is a solution to (4.2a). To obtain such an estimate assume the solution of (4.1) to be nonsingular. With our assumptions on Ω, problem (4.7) is W_2^2-regular. Hence we use standard arguments from [9]: duality estimates for problem (4.7) (Theorem II.1.2), Theorems IV.3.3, IV.3.5, and IV.4.2, finally for $\{\boldsymbol{u}, p\} \in \boldsymbol{H}_0^1 \cap H^d(\Omega)^n \times L_2/R \cap H^{d-1}(\Omega)$ with integer $d > 0$ and sufficiently small H we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_H\|_0^2 \leqslant C(\Omega, \nu, \tau, \|\boldsymbol{u}\|_{\ell_1+1}, \| p\|_{\ell_2})(H^{2(\ell_1+1)} + H^{2(\ell_2+1)}) \tag{4.9}$$

with $\ell_1 = \min(k_1, d-1)$, $\ell_2 = \min(k_2, d-1)$. This estimate provides us with the choice of scaling $h = \mathrm{O}(H^{1+1/\ell_1})$ to obtain an optimal accuracy $\mathrm{O}(h^{2\ell_1})$ in (4.3). However, in (4.11) a dependence of the constant C on ν and τ is implicit.

Corollary 4.4. *Let* $\{\boldsymbol{u}, p\} \in \boldsymbol{H}_0^1 \cap H^d(\Omega)^2 \times L_2/R \cap H^{d-1}(\Omega)$ *for some integer* $d > 0$ *be a solution to problem* (4.1) *and* $\{\boldsymbol{u}_H, p_H, \boldsymbol{u}_h, p_h\} \in \boldsymbol{H}_H \times Q_H \times \boldsymbol{H}_h \times Q_h$ *be a solution to problem* (4.2a), (4.2b) *then for sufficiently small H the following estimates hold*:

$$\| \boldsymbol{u}_h - \boldsymbol{u}_H \|_0^2 \leqslant C(\Omega, \nu, \tau, \| \boldsymbol{u} \|_{\ell_1+1}, \| p \|_{\ell_2})(H^{2(\ell_1+1)} + H^{2(\ell_2+1)})$$

with $\ell_1 = \min(k_1, d-1)$, $\ell_2 = \min(k_2, d-1)$.

Proof. The inequality

$$\| \boldsymbol{u}_h - \boldsymbol{u}_H \|_0^2 \leqslant 2(\| \boldsymbol{u} - \boldsymbol{u}_h \|_0^2 + \| \boldsymbol{u} - \boldsymbol{u}_H \|_0^2)$$

together with (4.9), Theorem 4.1, and assumption $h < H$ give the result. □

Another estimate on $\boldsymbol{u}_H - \boldsymbol{u}_h$ without explicitly involving solution of (4.1) is given by the following theorem.

Theorem 4.5. *For given* $\boldsymbol{g} \in \boldsymbol{H}^{-1}$ let $\{\boldsymbol{u}_H, p_H\} \in \boldsymbol{H}_H \times Q_H$ *and* $\{\boldsymbol{u}_h, p_h\} \in \boldsymbol{H}_h \times Q_h$ *be a solution to problem* (4.2a), (4.2b), *then the following estimate is valid*:

$$\| \boldsymbol{u}_h - \boldsymbol{u}_H \|_0^2 + \tau\nu \| \boldsymbol{u}_h - \boldsymbol{u}_H \|_1^2 \leqslant C(\Omega)\left((H^2 + \tau\nu) \inf_{\boldsymbol{v} \in \boldsymbol{H}_H} \| \boldsymbol{u}_h - \boldsymbol{v} \|_1^2 + (\tau\nu + \gamma_H) \inf_{q \in Q_H} \| p_h - q \|_0^2 \right). \tag{4.10}$$

Proof. For given $\{\boldsymbol{u}_h, p_h\}$ from (4.2a) let $\{\boldsymbol{u}_S, p_S\} \in \boldsymbol{V}_H \times Q_H$ be a solution to the Stokes problem

$$(\nabla \boldsymbol{u}_S, \nabla \boldsymbol{v}) - b(p_S, \boldsymbol{v}) = (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}) - b(p_h, \boldsymbol{v}),$$

$$b(q, \boldsymbol{u}_S) = 0, \quad \forall \{\boldsymbol{v}, q\} \in \boldsymbol{H}_H \times Q_H,$$

then the following estimates are valid:

$$\begin{aligned} \| \boldsymbol{u}_h - \boldsymbol{u}_S \|_1 &\leqslant c(\Omega)\left(\inf_{\boldsymbol{v} \in \boldsymbol{H}_H} \| \boldsymbol{u}_h - \boldsymbol{v} \|_1 + \inf_{q \in Q_H} \| p_h - q \|_0 \right), \\ \| \boldsymbol{u}_h - \boldsymbol{u}_S \|_0 &\leqslant c(\Omega) H \inf_{\boldsymbol{v} \in \boldsymbol{H}_H} \| \boldsymbol{u}_h - \boldsymbol{v} \|_1, \end{aligned} \tag{4.11}$$

Choose also $p_I = \arg\min_{q \in Q_H} \| p_h - q \|_0$.

Let us take in (4.2a) and (4.2b) $\boldsymbol{v} = \boldsymbol{u}_S - \boldsymbol{u}_H$ and substract (4.2b) from (4.2a), then we obtain

$$\begin{aligned} \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_0^2 + \tau\nu \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_1^2 &= a_\tau(\boldsymbol{u}_S - \boldsymbol{u}_h, \boldsymbol{u}_S - \boldsymbol{u}_H) + \tau b(p_h - p_I, \boldsymbol{u}_S - \boldsymbol{u}_H) \\ &\leqslant \| \boldsymbol{u}_S - \boldsymbol{u}_h \|_0^2 + \tau\nu \| \boldsymbol{u}_S - \boldsymbol{u}_h \|_1^2 + 2\tau \| p_h - p_I \|_0 \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_1 . \end{aligned} \tag{4.12}$$

Estimate the last term as

$$2\tau \| p_h - p_I \|_0 \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_1 \leqslant \tau\nu^{-1} \| p_h - p_I \|_0^2 + \tau\nu \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_1^2 \tag{4.13a}$$

and

$$2\tau \| p_h - p_I \|_0 \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_1 \leqslant \kappa^2\tau^2 H^{-2} \| p_h - p_I \|_0^2 + \tfrac{1}{2} \| \boldsymbol{u}_S - \boldsymbol{u}_H \|_0^2 . \tag{4.13b}$$

Now apply the triangle inequality, estimates (4.11), (4.12) and one of the estimates (4.13a) and (4.13b) and get (4.10). The theorem is proved.

5. Stability

We understand the stability of schemes (3.1)–(3.3) as a validation of some a priori estimates for finite element solutions obtained using these schemes. Week solution $\boldsymbol{u} \in L_2(0,T;\boldsymbol{H}_0^1) \cap L_\infty(0,T;\boldsymbol{L}^0)$ of problem (1.1)–(1.3) satisfies energy estimate

$$\|\boldsymbol{u}(t)\|_0^2 + 2\nu \int_0^t \|\boldsymbol{u}(s)\|^2 \, \mathrm{d}s \leqslant \|\boldsymbol{u}_0\|_0^2 + 2\int_0^t |(\boldsymbol{f}(s),\boldsymbol{u}(s))| \, \mathrm{d}s.$$

Further we will deduce some conditions that provide finite element solution with finite analogue of energy estimate (see also [25,26]).

The following theorem gives condition on τ_i that ensure (3.1) to be stable. This condition depends on $\boldsymbol{u}_h^i$, Corollary 5.2 shows that the condition can be strengthened and made depend only on the given data.

Theorem 5.1. *For $\Omega \in R^n$, $n = 1,2$ any natural $m \geqslant 0$ and any τ_i, $i = 0,\dots,m$ satisfying condition*

$$\tau_i \leqslant \min\left(\frac{h^n \nu}{8\kappa^2 \|\boldsymbol{u}_h^i\|_0^2}, \frac{h^2}{2\kappa^2 \nu}\right), \quad i = 0,1,\dots, \tag{5.1}$$

the following estimate holds for $\boldsymbol{u}_h^i$ determined from (3.1):

$$\|\boldsymbol{u}_h^{m+1}\|_0^2 + \frac{1}{4}\sum_{i=0}^{m} \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 + \nu \sum_{i=0}^{m} \tau_i \|\boldsymbol{u}_h^{i+1}\|_1^2 \leqslant \|\boldsymbol{u}_0\|_0^2 + \frac{2}{\nu}\sum_{i=0}^{m} \tau_i \|\boldsymbol{f}^{i+1}\|_{-1}^2, \tag{5.2}$$

Proof. Let us take in (3.1) $\boldsymbol{v} = 2\boldsymbol{u}_h^{i+1}$, we get the equality

$$\|\boldsymbol{u}_h^{i+1}\|_0^2 - \|\boldsymbol{u}_h^i\|_0^2 + \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 + 2\tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 = -2\tau_i N(\boldsymbol{u}_h^i;\boldsymbol{u}_h^i,\boldsymbol{u}_h^{i+1}) + 2\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{u}_h^{i+1}). \tag{5.3}$$

Right-hand side in (5.3) can be estimated as follows:

$$\begin{aligned}
-2\tau_i N(\boldsymbol{u}_h^i;\boldsymbol{u}_h^i,\boldsymbol{u}_h^{i+1}) + 2\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{u}_h^{i-1}) &= -2\tau_i N(\boldsymbol{u}_h^i;\boldsymbol{u}_h^i,\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i) + 2\tau_i(\boldsymbol{f}^{i+1},\boldsymbol{u}_h^{i+1}) \\
&\leqslant 2\tau_i\kappa h^{-n/2} \|\boldsymbol{u}_h^i\|_0 \|\boldsymbol{u}_h^i\|_1 \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0 + \tfrac{1}{2}\tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 + \frac{2\tau_i}{\nu}\|\boldsymbol{f}^{i+1}\|_{-1}^2 \\
&\leqslant 2\kappa^2\tau_i^2 h^{-n} \|\boldsymbol{u}_h^i\|_0^2 \|\boldsymbol{u}_h^i\|_1^2 + \tfrac{1}{2}\|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 + \tfrac{1}{2}\tau_i\nu \|\boldsymbol{u}_h^{i-1}\|_1^2 + \frac{2\tau_i}{\nu}\|\boldsymbol{f}^{i+1}\|_{-1}^2 .
\end{aligned}$$

From (5.3) and the last estimate we get

$$\|\boldsymbol{u}_h^{i+1}\|_0^2 + \tfrac{1}{2}\|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 + \tfrac{3}{2}\tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 - 4\kappa^2\tau_i^2 h^{-n}\|\boldsymbol{u}_h^i\|_0^2\|\boldsymbol{u}_h^i\|_1^2 \leqslant \|\boldsymbol{u}_h^i\|_0^2 + \frac{2\tau_i}{\nu}\|\boldsymbol{f}^{i+1}\|_{-1}^2 . \tag{5.4}$$

By virtue of condition (5.1), for τ_i the following estimates are valid:

$$\begin{aligned}
\tfrac{3}{2}\tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 - 4\kappa^2\tau_i^2 h^{-n} \|\boldsymbol{u}_h^i\|_0^2\|\boldsymbol{u}_h^i\|_1^2 &\geqslant \tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 + \tfrac{1}{2}\tau_i\nu(\|\boldsymbol{u}_h^{i+1}\|_1^2 - \|\boldsymbol{u}_h^i\|_1^2) \\
&\geqslant \tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 - \tfrac{1}{2}\nu\tau_i \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_1^2 \geqslant \tau_i\nu \|\boldsymbol{u}_h^{i+1}\|_1^2 - \tfrac{1}{4}\|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 .
\end{aligned}$$

Now we get from (5.4)

$$\|\boldsymbol{u}_h^{i+1}\|_0^2 + \tfrac{1}{4}\|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_0^2 + \nu\tau_i \|\boldsymbol{u}_h^{i+1}\|_1^2 \leqslant \|\boldsymbol{u}_h^i\|_0^2 + \frac{2\tau_i}{\nu}\|\boldsymbol{f}^{i+1}\|_{-1}^2 . \tag{5.5}$$

Taking for $i=0,\dots,m$ a sum of (5.5), we obtain (5.2) and prove the theorem.

Corollary 5.2. *There exist some constant $c=c(\Omega,\boldsymbol{f}(t),\boldsymbol{u}_0)>0$ such that for any set of τ_i, $i=0,\dots,m$, satisfying condition*

$$\tau_i \leqslant \min\left(c\,h^n v^2, \frac{h^2}{2\kappa^2 v}\right), \quad i=0,1,\dots, \tag{5.6}$$

and $\boldsymbol{u}_h^i$ determined from (3.1) *estimate* (5.2) *holds with any natural $m\geqslant 0$.*

Proof. Estimate (5.2) for any set of τ_i satisfying (5.1) provides $\|\boldsymbol{u}_h^i\|_0^2 \leqslant c_1(\boldsymbol{u}_0,\boldsymbol{f}(t))v^{-1}$. Setting in (5.6) $c=(8\kappa^2 c_1)^{-1}$, we prove the corollary.

Note that Corollary 5.2 recovers asymptotically the same condition on the constant step τ as Lemma 5.3 from [25]. However, if in a particular problem $\|\boldsymbol{u}_h^i\|_0$ depends on v in some more advantageous way, then the condition on τ can be weakened; this is the situation which we have sometime in practice (see [23] for calculations of critical (for stability) τ for scheme (3.1) in the case of one substantially nonlinear and unsteady flow).

The following theorem sharpens the estimate on τ with respect to v. The theorem requires that function $\boldsymbol{f}$ satisfies $\|\boldsymbol{f}(t)\|_{L_2(G)} \leqslant c < \infty$ for some $T>0$, $G=(0,T)\times\Omega$.

Theorem 5.3. *For any $T>0$ there exist some constant $c=c(T,\Omega,\boldsymbol{f}(t),\boldsymbol{u}^0)>0$ such that for any τ satisfying the condition*

$$\tau \leqslant \min\left(ch^n v, \frac{h^2}{2\kappa^2 v}\right), \quad i=0,1,\dots, \tag{5.7}$$

and for any natural $m\geqslant 0$ such that $m\tau\leqslant T$ the following estimate holds:

$$\|\boldsymbol{u}_h^{m+1}\|_0^2 + \frac{1}{4}\sum_{i=0}^{m}\|\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i\|_0^2 + v\sum_{i=0}^{m}\tau_i\|\boldsymbol{u}_h^{i+1}\|_1^2 \leqslant 2\exp(t)\left(\|\boldsymbol{u}_0\|_0^2 + \int_0^t \|\boldsymbol{f}\|_0^2\right) \tag{5.8}$$

with $t=m\tau$.

Proof. Fix T and choose τ satisfying (5.8) with

$$c=\frac{1}{8\kappa^2}[2\exp(T)(\|\boldsymbol{u}_0\|_0^2 + \|\boldsymbol{f}\|_{L_2(G)}^2)]^{-1}. \tag{5.9}$$

Let us prove by induction with respect to m the following estimate:

$$\|\boldsymbol{u}_h^{m+1}\|_0^2 + \frac{1}{4}\sum_{i=0}^{m}\|\boldsymbol{u}_h^{m+1}-\boldsymbol{u}_h^m\|_0^2 + \tau v\sum_{i=0}^{m}\|\boldsymbol{u}_h^{m+1}\|_1^2 \leqslant (1-\tau)^{-m}\left(\|\boldsymbol{u}^0\|_0^2 + \sum_{i=0}^{m}\tau\|\boldsymbol{f}^{i+1}\|_0^2\right). \tag{5.10}$$

Consider the case $m=0$. Indeed from (5.9) and (5.7) we have $\tau\leqslant h^n v(8\kappa^2\|\boldsymbol{u}\|_0^2)^{-1}$ and condition (5.1) for $\tau_0=\tau$ is fulfilled. Similar to the proof of Theorem 5.1 let us estimate right-hand side of (5.1) with $i=0$. However, now we apply the inequality $2\tau(\boldsymbol{f}^0,\boldsymbol{u}_h^1)\leqslant\tau\|\boldsymbol{u}_h^1\|_0^2+\tau\|\boldsymbol{f}^0\|_0^2$ instead of $2\tau(\boldsymbol{f}^0,\boldsymbol{u}_h^1)\leqslant\frac{1}{2}\tau v\|\boldsymbol{u}_h^1\|_1^2+(2/v)\tau\|\boldsymbol{f}^0\|_{-1}^2$.

Finally, we get

$$\| u_h^1 \|_0^2 + \tfrac{1}{4} \| u_h^1 - u^0 \|_0^2 + \tau\nu \| u_h^1 \|_1^2 \leqslant \| u^0 \|_0^2 + \tau \| u_h^1 \|_0^2 + \tau \| f^0 \|_0^2$$

that implies

$$\| u_h^1 \|_0^2 + \tfrac{1}{4} \| u_h^1 - u^0 \|_0^2 + \tau\nu \| u_h^1 \|_1^2 \leqslant (1-\tau)^{-1}(\| u^0 \|_0^2 + \tau \| f^0 \|_0^2).$$

Thus, the "trivial" case of $m = 0$ is checked.

Suppose now that (5.10) is proved for all $m = 0, 1, \ldots, k$. Let us prove it for $m = k + 1$, assuming $(k+1)\tau \leqslant T$. From inductive hypothesis we have (5.10) for $m = k$, hence

$$\| u_h^{k+1} \|_0^2 \leqslant (1-\tau)^{-k}(\| u^0 \|_0^2 + \| f \|_{L_2(G)}^2)$$

and condition (5.1) for $\tau_k = \tau$ is valid. With the same arguments as in the proof of Theorem 5.1 and another estimate for $2\tau(f^k, u_h^{k+1})$ we get

$$\| u_h^{k+1} \|_0^2 + \tfrac{1}{4} \| u_h^{k+1} - u_h^k \|_0^2 + \tau\nu \| u_h^{k+1} \|_1^2 \leqslant \| u_h^k \|_0^2 + \tau \| u_h^{k+1} \|_0^2 + \tau \| f^k \|_0^2,$$

and with inductive hypothesis

$$\begin{aligned}
&\| u_h^{k+1} \|_0^2 + \tfrac{1}{4} \| u_h^{k+1} - u_h^k \|_0^2 + \tau\nu \| u_h^{k+1} \|_1^2 \\
&\quad \leqslant (1-\tau)^{-1}\left[(1-\tau)^{-(k-1)}\left(\| u^0 \|_0^2 + \sum_{i=0}^{k-1} \tau \| f^{i+1} \|_0^2\right) - \frac{1}{4}\sum_{i=1}^{k} \| u_h^k - u_h^{k-1} \|_0^2 - \nu \sum_{i=1}^{k} \tau_i \| u_h^k \|_1^2\right] \\
&\qquad + (1-\tau)^{-1}\tau \| f^k \|_0^2 \leqslant (1-\tau)^{-k}\left(\| u^0 \|_0^2 + \sum_{i=0}^{k} \tau \| f^{i+1} \|_0^2\right) \\
&\qquad - \frac{1}{4}\sum_{i=1}^{k} \| u_h^k - u_h^{k-1} \|_0^2 - \nu \sum_{i=1}^{k} \tau_i \| u_h^k \|_1^2 .
\end{aligned}$$

Thus, (5.10) is proved for all $m{:}m\tau \leqslant T$. Estimate (5.8) follows from (5.10) and obvious relations:

$$(1-\tau)^{-m} = \left(1 - \frac{t}{m}\right)^{-m} < \exp(t)(1-\tau)^{-1} \leqslant 2\exp t,$$

$$\int_0^t \| f \|_0^2 = \sum_{i=0}^{m} \tau \| f^{i+1} \|_0^2 .$$

The theorem is proved. □

Note that Theorem 5.3 permits the exponential growth of $\| u_h \|$; however, this is an admissible assumption in the stability theory for stiff systems (see, e.g. [7]).

Now we prove a stability result for two-level scheme (3.2) with $\tau_i = \tau,\ i = 0, 1, \ldots$.

Theorem 5.4. *Assume that* $H = o(h^{(1+s)/2})$. *Then there exists some real positive constant* $c_1(\nu, \tau, f, u_0, T)$ *such that for any real* $h > 0$ *satisfying*

$$h \leqslant c_1(\nu, \tau, f, u_0, T), \tag{5.11}$$

the following estimate holds for u_h^i, $i=0,\dots,m$ *determined from* (3.2a) and (3.2b):

$$\| u_h^{m+1} \|_0^2 + \frac{1}{4}\sum_{i=0}^{m} \| u_h^{i+1} - u_h^i \|_0^2 + \nu \sum_{i=0}^{m} \tau \| u_h^{i+1} \|_1^2 \leqslant ct + \| u_0 \|_0^2 + \frac{2}{\nu}\sum_{i=0}^{m} \tau \| f^{i+1} \|_{-1}^2 \tag{5.12}$$

with $t=(m+1)\tau$, $t\leqslant T$ *and any integer* $m\geqslant 0$. *Moreover*

$$\sum_{i=0}^{m} \tau \| p_h^{i+1} \|_0^2 \leqslant H^{-s} c_2(\nu,\Omega,f,u_0,s,T) \tag{5.13}$$

with some real positive constant $c_2(\nu,\Omega,f,u_0,s,T)$.

Proof. Let us take in (3.2b) $v=2u_h^{i+1}$. We get the equality

$$\begin{aligned} &\| u_h^{i+1} \|_0^2 - \| u_h^i \|_0^2 + \| u_h^{i+1} - u_h^i \|_0^2 + 2\tau\nu \| u_h^{i+1} \|_1^2 \\ &= -2\tau N(u_H^{i+1}; u_H^{i+1}, u_h^{i+1}) + 2\tau(f^{i+1}, u_h^{i+1}). \end{aligned} \tag{5.14}$$

The right-hand side in (5.14) can be estimated as follows:

$$\begin{aligned} &-2\tau N(u_H^{i+1}; u_H^{i+1}, u_h^{i+1}) + 2\tau(f^{i+1}, u_h^{i+1}) \\ &\quad = 2\tau N(u_h^{i+1}; u_h^{i+1} - u_H^{i+1}, u_h^{i+1}) - 2\tau N(u_h^{i+1} - u_H^{i+1}; u_h^{i+1} - u_H^{i+1}, u_h^{i+1}) \\ &\qquad + 2\tau(f^{i+1}, u_h^{i+1}) \leqslant c(s)\tau \| u_h^{i+1} - u_H^{i+1} \|_s \| u_h^{i+1} \|_1^2 \\ &\qquad + c(s)\tau \| u_h^{i+1} - u_H^{i+1} \|_s \| u_h^{i+1} - u_H^{i+1} \|_1 \| u_h^{i+1} \|_1 + \tfrac{1}{2}\tau\nu \| u_h^{i+1} \|_1^2 + \frac{2\tau}{\nu} \| f^{i+1} \|_{-1}^2 \\ &\quad \leqslant c(s)\tau h^{-s} \| u_h^{i+1} - u_H^{i+1} \|_0 \| u_h^{i+1} \|_1^2 \\ &\qquad + c(s)\tau h^{-1-s} \| u_h^{i+1} - u_H^{i+1} \|_0^2 \| u_h^{i+1} \|_1 + \tfrac{1}{2}\tau\nu \| u_h^{i+1} \|_1^2 + \frac{2\tau}{\nu} \| f^{i+1} \|_{-1}^2 . \end{aligned} \tag{5.15}$$

Denote by $c_3(\nu,\tau)$ a positive constant from the estimate

$$\| u \|_1 + \| p \|_0 \leqslant c_3(\nu,\tau) \| g \|_0, \tag{5.16}$$

for solution of (4.1) with given $g\in L_2(\Omega)^2$. Let us consider

$$a = c_3(\nu,\tau)\left[\left(cT + \| u_h^0 \|_0^2 + \frac{2}{\nu}\max_{[0,T]} \| f \|_{-1}^2\right)^{1/2} + \| f \|_{L_2(G)}\right].$$

Then from Corollary 4.4 (with $\ell_1=\ell_2=0$) it follows that

$$c(s)h^{-s} \| u_h^1 - u_H^1 \|_0 \leqslant C(\Omega,\nu,\tau,a,a)^{1/2} H \tag{5.17}$$

with $C(\Omega,\nu,\tau,\cdot,\cdot)$ from Corollary 4.4.

Further we prove (5.12) by induction with respect to m. Consider the case $m=0$. Indeed (5.17) and relation $H=o(h^{(1+s)/2})$ provide us with sufficiently small h such that $c(s)h^{-s} \| u_h^1 - u_H^1 \|_0 \leqslant \nu/4$ and $c(s)h^{-2-2s} \| u_h^1 - u_H^1 \|_0^4 \leqslant \nu/4$. Thus,

$$c(s)\tau h^{-s} \| u_h^1 - u_H^1 \|_0 \| u_h^1 \|_1^2 \leqslant \tfrac{1}{4}\tau\nu \| u_h^1 \|_1^2 \tag{5.18}$$

and

$$c(s)\tau h^{-1-s}\,\|\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_H^{i+1}\|_0^2\|\boldsymbol{u}_h^1\|_1 \leqslant \frac{c(s)}{2}\tau(1+h^{-2-2s}\,\|\boldsymbol{u}_h^1-\boldsymbol{u}_H^1\|_0^4\|\boldsymbol{u}_h^1\|_1^2)$$
$$\leqslant \tfrac{1}{2}\tau(c(s)+\tfrac{1}{2}\nu\,\|\boldsymbol{u}_h^1\|_1^2). \tag{5.19}$$

From (5.14), (5.15) with $i=0$, (5.18), and (5.19) we get (5.12) for $m=0$. The case of $m=0$ is checked.

Assuming that (5.12) is valid for some $m=j-1$, we prove this estimate for $m=j$ ($(m+1)\tau\leqslant T$) in the same way as for $m=0$: starting with (5.15) for $i=j$ and checking with the help of inductive hypothesis

$$c(s)h^{-s}\,\|\boldsymbol{u}_h^j-\boldsymbol{u}_H^j\|_0 \leqslant C(\Omega,\nu,\tau,a,a)^{1/2}H. \tag{5.20}$$

Then in the similar way as (5.18), (5.19) for sufficiently small h it follows

$$c(s)\tau\, h^{-s}\,\|\boldsymbol{u}_h^{j+1}-\boldsymbol{u}_H^{j+1}\|_0\|\boldsymbol{u}_h^{j+1}\|_1^2 \leqslant \tfrac{1}{4}\tau\nu\,\|\boldsymbol{u}_h^{j+1}\|_1^2$$

and

$$c(s)\tau h^{-1-s}\,\|\boldsymbol{u}_h^{j+1}-\boldsymbol{u}_H^{j+1}\|_0^2\|\boldsymbol{u}_h^{j+1}\|_1 \leqslant \tfrac{1}{2}\tau(c(s)+\tfrac{1}{2}\nu\,\|\boldsymbol{u}_h^{j+1}\|_1^2).$$

Now we obtain from (5.14)

$$\|\boldsymbol{u}_h^{j+1}\|_0^2+\|\boldsymbol{u}_h^{j+1}-\boldsymbol{u}_h^j\|_0^2+\tau\nu\,\|\boldsymbol{u}_h^{j+1}\|_1^2 \leqslant c\tau+\|\boldsymbol{u}_h^j\|_0^2+2\frac{\tau}{\nu}\,\|\boldsymbol{f}^{j+1}\|_{-1}^2\,. \tag{5.21}$$

Taking a sum of (5.21) and (5.12) with $m=j-1$ we obtain (5.12) for $m=j$, hence we complete the inductive step and prove (5.12) for any m satisfying $(m+1)\tau<T$.

To obtain an estimate on pressure function, let us rewrite (3.2b) as

$$\tau b(p_h^{i+1},\boldsymbol{v})=\tau\nu a(\boldsymbol{u}_h^{i+1},\boldsymbol{v})+(\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i,\boldsymbol{v})+\tau(f^{i+1},\boldsymbol{v})+\tau N(\boldsymbol{u}_H^{i+1},\boldsymbol{u}_H^{i+1},\boldsymbol{v}).$$

with arbitrary $\boldsymbol{v}\in\boldsymbol{H}_h$. Hence the stability hypothesis provides us with estimate

$$\|p_h^{i+1}\|_0^2 \leqslant c(\Omega,\nu)\left(\|\boldsymbol{u}_h^{i+1}\|_1^2+\left\|\frac{\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i}{\tau}\right\|_{-1}^2+\|f^{i+1}\|_{-1}^2+c(s)H^{-s}\,\|\boldsymbol{u}_H^{i+1}\|_0^2\|\boldsymbol{u}_H^{i+1}\|_1^2\right)$$

and

$$\sum_{i=0}^m\tau\,\|p_h^{i+1}\|_0^2 \leqslant c(\Omega,\nu)\left(\sum_{i=0}^m\tau\,\|\boldsymbol{u}_h^{i+1}\|_1^2+\sum_{i=0}^m\tau\,\|f^{i+1}\|_{-1}^2\right.$$
$$\left.+\sum_{i=0}^m\tau\left\|\frac{\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i}{\tau}\right\|_{-1}^2+c(s)H^{-s}\max_i(\|\boldsymbol{u}_H^{i+1}\|_0^2)\sum_{i=0}^m\tau\,\|\boldsymbol{u}_H^{i+1}\|_1^2\right). \tag{5.22}$$

The first term in estimate (5.22) is bounded due to (5.12), the second depends only on given data. Estimate $\sum_{i=0}^m\tau\,\|\tau^{-1}(\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i)\|_{-1}^2 \leqslant cH^{-s}$ is proved similar to Lemma III.4.6 in [25].

To obtain estimates on $\boldsymbol{u}_H$ let us write down (3.2a) with $\boldsymbol{v}=\boldsymbol{u}_H^{i+1}$ (note that (5.12) provides us with estimate on $\boldsymbol{u}_h$ but not on $\boldsymbol{u}_H$):

$$(\boldsymbol{u}_H^{i+1},\boldsymbol{u}_H^{i+1})+\tau\nu a(\boldsymbol{u}_H^{i+1},\boldsymbol{u}_H^{i+1})=(\boldsymbol{u}_h^i,\boldsymbol{u}_H^{i+1})+\tau(f^{i+1},\boldsymbol{u}_H^{i+1}). \tag{5.23}$$

From (5.23) we readily get for $i = 0,\dots,m$

$$\begin{aligned} &\| \boldsymbol{u}_H^{i+1} \|_0^2 + \tau\nu \| \boldsymbol{u}_H^{i+1} \|_1^2 \leqslant \| \boldsymbol{u}_h^i \|_0^2 + \frac{\tau}{\nu} \| \boldsymbol{f}^{i+1} \|_{-1}^2, \\ &\| \boldsymbol{u}_H^{i+1} \|_0^2 + \tau\nu \| \boldsymbol{u}_H^{i+1} \|_1^2 \leqslant \| \boldsymbol{u}_H^i \|_0^2 + \frac{\tau}{\nu} \| \boldsymbol{f}^{i+1} \|_{-1}^2 + 2(\boldsymbol{u}_h^i - \boldsymbol{u}_H^i, \boldsymbol{u}_H^{i+1}); \end{aligned} \tag{5.24}$$

in the second inequality, $\boldsymbol{u}_H^i$ should be replaced by $\boldsymbol{u}_h^i$ for $i = 0$.

The first inequality in (5.24) and (5.12) gives an estimate on $K = \max_i (\| \boldsymbol{u}_H^{i+1} \|_0^2) \leqslant c(\Omega, \boldsymbol{u}_0, \boldsymbol{f}, \nu, T)$. To obtain a bound for $\sum_{i=0}^m \tau \| \boldsymbol{u}_H^{i+1} \|_1^2$ in (5.22) let us take a sum of the second inequality in (5.24) for $i = 0,\dots,m$:

$$\sum_{i=0}^m \tau \| \boldsymbol{u}_H^{i+1} \|_1^2 \leqslant \| \boldsymbol{u}_h^0 \|_0^2 + \frac{1}{\nu} \sum_{i=0}^m \tau \| \boldsymbol{f}^{i+1} \|_{-1}^2 + 2 \sum_{i=1}^m (\boldsymbol{u}_h^i - \boldsymbol{u}_H^i, \boldsymbol{u}_H^{i+1}) \tag{5.25}$$

and act as follows. For sufficiently small h, using (5.20), we obtain

$$\begin{aligned} \sum_{i=1}^m (\boldsymbol{u}_h^i - \boldsymbol{u}_H^i, \boldsymbol{u}_H^{i+1}) &\leqslant \max_i (\| \boldsymbol{u}_H^{i+1} \|_0) \sum_{i=1}^m \| \boldsymbol{u}_h^i - \boldsymbol{u}_H^i \|_0 \\ &\leqslant \sqrt{K} C(\Omega, \nu, \tau, a, a)^{1/2} \sum_{i=1}^m H \leqslant c(\Omega, \boldsymbol{u}_0, \boldsymbol{f}, \nu, T). \end{aligned} \tag{5.26}$$

Estimate (5.26) together with (5.25) gives

$$\sum_{i=0}^m \tau \| \boldsymbol{u}_H^{i+1} \|_1^2 \leqslant c_3(\Omega, \boldsymbol{u}_0, \boldsymbol{f}, \nu, T, s).$$

Thus all the terms in a right-hand side of (5.22) are estimated and the desired inequality (5.13) follows.

Remark 5.5. The crucial point in the proof of the theorem is obtaining an estimate on $\| \boldsymbol{u}_h - \boldsymbol{u}_H \|$. Corollary 4.4 shows that this estimate improves for equal order interpolation or high-order finite element schemes, if we assume an extra regularity of the solution obtained. Hence, in this case the stability conditions could be weakened. However, the lack of information about high-order norms in our stability estimates does not permit us to state precise results on this matter.

Compare condition (5.6) for scheme (3.1) with condition (5.11) for two-level scheme (3.2). Scheme (3.1) imposes restriction on τ, while scheme (3.2) imposes restriction on H. Although extensive calculations are needed to check whether the condition on H is restrictive indeed, the latest fact is of certain theoretical interest.

Finally we note that scheme (3.3) is unconditionally stable.

6. Convergence

Let us fix some $T > 0$ and for a sequence of $\boldsymbol{u}_h^i$, $i = 1,\dots,m$, $m\tau = T$ defined from any of schemes (3.1)–(3.3) denoted by $\boldsymbol{u}_h = \boldsymbol{u}_h(t) : [0, T] \to \boldsymbol{H}_h$, the following function:

$$\boldsymbol{u}_h(t) = \boldsymbol{u}_h^i \quad \text{for } t \in [(i-1)\tau, i\tau).$$

For scheme (3.3) function $\boldsymbol{u}_h(t)$ converges to solution $\boldsymbol{u}$ of (1.1)–(1.3) strongly in $L_2(G)$, $G=(0,T)\times\Omega$ and weakly in $L_2(0,T;\boldsymbol{H}_0^1)$. The same is valid for scheme (3.1) with τ,h satisfying stability condition (5.6).

Here we prove

Theorem 6.1. *Assume $\Omega\in R^2$. Let $\boldsymbol{u}_h^i$, $i=0,\dots,m$, from $\boldsymbol{H}_h$ recovered via two-level scheme* (3.2) *then*

$$\boldsymbol{u}_h\to\boldsymbol{u} \text{ strongly in } L_2(G) \text{ for } h,\tau\to 0,$$

$$\boldsymbol{u}_h\to\boldsymbol{u} \text{ weakly in } L_2(0,T;\boldsymbol{H}_0^1) \text{ for } h,\tau\to 0,$$

if condition (5.11) *and assumptions of Theorem* 5.4 *are satisfied. Here $\boldsymbol{u}$ is a unique weak solution to problem* (1.1)–(1.3).

Proof. The proof is close to those of Temam [25, Theorem III.5.4] for scheme 3.3. And for some omitted technical details we refer to the [25].

Along with $\boldsymbol{u}_h$ consider

$$\boldsymbol{w}_h(t)=\frac{(i-1)\tau-t}{(i-1)\tau-i\tau}\boldsymbol{u}_h^i+\frac{t-i\tau}{(i-1)\tau-i\tau}\boldsymbol{u}_h^{i-1}\quad\text{for } t\in[(i-1)\tau,i\tau],$$

and $\boldsymbol{w}_h(t)=0$ for $t\notin[0,T]$.

Since condition (5.11) is assumed to be satisfied, from estimate (5.12) we get

$$\|\boldsymbol{u}_h\|_{L_\infty(0,T;L_2(\Omega)^2)}\leqslant c,\qquad \|\boldsymbol{w}_h\|_{L_\infty(0,T;L_2(\Omega)^2)}\leqslant c,$$

and for some subsequence still denoted by $h,\tau(\to 0)$

$$\boldsymbol{u}_h\to\boldsymbol{u} \text{ weakly in } L_2(0,T;\boldsymbol{H}_0^1),\qquad \boldsymbol{w}_h\to\boldsymbol{w} \text{ weakly in } L_2(0,T;\boldsymbol{H}_0^1) \tag{6.1}$$

with some $\boldsymbol{u},\boldsymbol{w}\in L_\infty(0,T;L_2(\Omega)^2)\cap L_2(0,T;\boldsymbol{H}_0^1)$. Moreover [25],

$$\boldsymbol{w}_h\to\boldsymbol{w} \text{ strongly in } L_2(G). \tag{6.2}$$

Further, from equality

$$\|\boldsymbol{u}_h-\boldsymbol{w}_h\|_{L_2(G)}=\sqrt{\frac{\tau}{3}}\left(\sum_{i=0}^{m}\|\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i\|_0^2\right)^{1/2}$$

estimate (5.12), (6.1) and (6.2) we get

$$\boldsymbol{u}=\boldsymbol{w},\qquad \boldsymbol{u}_h\to\boldsymbol{u} \text{ strongly in } L_2(G). \tag{6.3}$$

Now let us set

$$\boldsymbol{u}_H(t)=\boldsymbol{u}_H^i\quad\text{for } t\in[(i-1)\tau,i\tau).$$

From (4.10) and hypothesis (H1) we get

$$\|\boldsymbol{u}_h-\boldsymbol{u}_H\|_{L_2(G)}^2\leqslant c(\nu)\left[(H^2+\tau)\sum_{i=0}^{m}\tau\|\boldsymbol{u}_h^i\|_1^2+\tau\sum_{i=0}^{m}\tau\|p^i\|_0^2\right] \tag{6.4a}$$

and

$$\tau \, \| \boldsymbol{u}_h - \boldsymbol{u}_H \|^2_{L_2(0,T;\boldsymbol{H}^1_0)} \leqslant c(\nu) \left[(H^2+\tau) \sum_{i=0}^{m} \tau \, \| \boldsymbol{u}^i_h \|^2_1 + \tau \sum_{i=0}^{m} \tau \, \| p^i \|^2_0 \right]. \tag{6.4b}$$

Hence we imply (5.12), (5.13) with sufficiently small s, (6.1), (6.3) and obtain

$$\boldsymbol{u}_H \to \boldsymbol{u} \text{ strongly in } L_2(G), \quad \boldsymbol{u}_H \to \boldsymbol{u} \text{ weakly in } L_2(0,T;\boldsymbol{H}^1_0). \tag{6.5}$$

To prove that $\boldsymbol{u}$ is a weak solution of (1.1)–(1.3) it is sufficient to note that (3.2b) provides

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_h(t), \boldsymbol{v}) + a(\boldsymbol{u}_h(t), \boldsymbol{v}) + N(\boldsymbol{u}_H(t); \boldsymbol{u}_H(t), \boldsymbol{v}) = (\boldsymbol{f}_h(t), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

and imply (6.1)–(6.3),(6.5) and arguments of Lemma III.5.9 from [25] in a straightforward way. The theorem is proved. □

Remark 6.2. The difficulty in proving Theorem 6.1 for $\Omega \subset R^3$ is caused by the impossibility of taking sufficiently small s in (5.3). Dependence of constant c_2 in Theorem 5.3 on τ is generally unknown. Therefore pressure terms in (6.4a) and (6.4b) failed to be properly estimated.

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