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ON ERROR ANALYSIS FOR THE 3D NAVIER–STOKES EQUATIONS IN VELOCITY-VORTICITY-HELICITY FORM*

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Abstract. We present a rigorous numerical analysis and computational tests for the Galerkin finite element discretization of the velocity-vorticity-helicity formulation of the equilibrium Navier–Stokes equations (NSEs). This formulation, recently derived by the authors, is the first NSE formulation that directly solves for helicity and the first velocity-vorticity formulation to naturally enforce incompressibility of the vorticity, and preliminary computations confirm its potential. We present a numerical scheme; prove stability, existence of solutions, uniqueness under a small data condition, and convergence; and provide numerical experiments to confirm the theory and illustrate the effectiveness of the scheme on a benchmark problem.

Key words. Navier–Stokes, velocity-vorticity, helicity, finite element method, nonhomogeneous boundary conditions

AMS subject classifications. 35Q30, 65N30, 76D05

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1. Introduction.

Incompressible viscous flows of a Newtonian fluid are modeled by the system of the Navier–Stokes equations (NSEs), which read as follows: Given a bounded, connected domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth Lipschitz boundary $\partial\Omega$, the simulation time T, and a force field $\mathbf{f} : Q \to \mathbb{R}^3$ (henceforth $Q := (0, T) \times \Omega$), find a velocity field $\mathbf{u} : Q \to \mathbb{R}^3$ and a pressure field $p : Q \to \mathbb{R}$ such that

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } Q, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \overline{Q}, \tag{1.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \tag{1.3}$$

$$\mathbf{u} = \boldsymbol{\phi} \quad \text{on } (0, T) \times \partial\Omega, \tag{1.4}$$

where $\nu > 0$ is the kinematic viscosity coefficient, u_0 is the divergence-free initial velocity, and the function $\boldsymbol{\phi}$ in the Dirichlet boundary condition satisfies $\int_\Omega \boldsymbol{\phi} \cdot \mathbf{n} = 0$ $\forall\, t \in (0, T)$.

In [22], the authors derived a new formulation of the NSEs, called the velocity-vorticity-helicity (VVH) formulation (1.3)–(1.9). Initial testing of VVH has shown great promise, and there are several novel properties of the formulation that warrant further development and testing. First, VVH is a velocity-vorticity system, which can provide more accurate solutions than velocity-pressure systems [8, 10, 11, 18, 19, 24, 26, 28, 27]. Typically such an improvement in accuracy comes at a cost, but in [22] an efficient iterative scheme for VVH is devised that performed very well on initial

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tests. Second, VVH directly solves for helical density $hel := \mathbf{u} \cdot (\nabla \times \mathbf{u})$, which is related to helicity by $H = \int_\Omega hel \; d\mathbf{x}$. Helicity is physically interpreted to be the degree to which a flow's vortices are tangled and intertwined, and it is believed to be of fundamental importance in understanding turbulence because it is a conserved quantity describing rotation, it cascades jointly with energy through the inertial range, and it plays a role in vortex breakdown and determination of a flow's reflectional symmetry [1, 3, 6, 9, 17, 20]. VVH is the first formulation that directly solves for helicity in any form. Third, VVH naturally and explicitly enforces incompressibility of the vorticity by identifying helical density as a Lagrange multiplier corresponding to the constraint, similar to the relationship of pressure and mass conservation in the velocity-pressure formulation. VVH is the first velocity-vorticity formulation to naturally enforce this fundamental constraint.

The VVH system can be derived as follows. Taking the curl of (1.1)–(1.2) and using vector identities (see [22]), we get the vorticity equations

$$\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + 2D(\mathbf{w})\mathbf{u} - \nabla hel = \nabla \times \mathbf{f} \quad \text{in } Q, \tag{1.5}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \overline{Q}, \tag{1.6}$$

$$\mathbf{w}|_{t=0} = \nabla \times \mathbf{u}_0 \quad \text{in } \Omega, \tag{1.7}$$

where $\mathbf{w} := \nabla\times\mathbf{u}$ is the vorticity, $hel := \mathbf{u}\cdot\mathbf{w}$ the helical density, and $D(\mathbf{w}) := \frac{1}{2}(\nabla\mathbf{w} + [\nabla\mathbf{w}]^T)$ the symmetric part of the vorticity gradient tensor. There are several options for completing the above equations with boundary conditions and velocity equations; cf. [22]. In this paper we consider the Dirichlet (kinematic) vorticity conditions,

$$\mathbf{w} = \boldsymbol{\psi} \quad \text{on } (0,T) \times \partial\Omega, \tag{1.8}$$

where $\boldsymbol{\psi} := \nabla \times \mathbf{u}|_{(0,T)\times\partial\Omega}$, and the *rotation* form of the NSEs,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla P = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } Q, \tag{1.9}$$

subject to (1.3)–(1.4), where $P = \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + p$ is the Bernoulli pressure variable.

Several time discretization algorithms for the coupled system (1.3)–(1.9) were developed and numerically tested in [22]. However, a rigorous analysis of this formulation and related discretizations has yet to be performed, and, to the best of our knowledge, no such analysis has been performed for any velocity-vorticity method; see Remark 4.9 for discussion of the relevance of the present analysis for different velocity-vorticity formulations. Even for the case of equilibrium flow, analyzing such methods remains a difficult task, and to the best of our knowledge also has not been done. Thus, we study the Galerkin finite element formulation of VVH for the equilibrium NSEs. The specific difficulties for numerical analysis of velocity-vorticity formulations include the following: (i) Most of the discrete velocity-vorticity formulations are lacking a basic energy estimate for velocity, typically supplied by the momentum equation but not by (discrete) Poisson velocity equation ($-\Delta\mathbf{u} = \mathbf{w}$) or the Cauchy–Riemann equations which are commonly used to supplement vorticity equations.[1] Following [22], this difficulty is overcome by "closing" (1.5)–(1.7) with the rotation form of the

[1] While in the continuous setting the formulations remain equivalent to (1.1)–(1.4), this is not the case for discretized problems, resulting in the lack of direct arguments to show energy estimate for *discrete* velocities.

momentum equation (1.9). (ii) Stability in the "energy"-type norm for discrete vorticity calls for stability of $\mathbf{u}$ in H^2, which is not obvious to show for *discrete* velocities. For this reason, in this paper we check stability for discrete vorticity in the L^2 norm. (iii) Finally, the algorithmic bottleneck of velocity-vorticity approaches—the definition of vorticity boundary conditions—is also a challenge for numerical analysis (we address this issue in several instances throughout the paper).

Our analysis is for the case of homogeneous Dirichlet velocity boundary conditions (which can be generalized in the usual way [25]), and for the vorticity boundary condition we study the use of an appropriate interpolant of the curl of (1) continuous and (2) discrete velocities. Although case (1) is not always realistic, a complete analysis of the more practical case (2) was found intractable by the authors due to unresolved regularity questions of the interpolant of curl of the discrete velocity on the boundary (see section 4). Still, analysis of case (1) resolves significant technical difficulties not present in the unrealistic cases of periodic or homogeneous Dirichlet velocity and vorticity boundary conditions, and is therefore an important step toward completing an analysis of case (2).

2. Preliminaries.

We will denote the $L^2(\Omega)$ norm and inner product by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The $H^k(\Omega)$ norms will be denoted by $\|\cdot\|_k$. All other norms will be clearly labeled with subscripts. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ which has zero trace on the boundary, and by $L_0^2(\Omega)$ the zero-mean subspace of $L^2(\Omega)$,

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \ : \ \int_\Omega q \, d\mathbf{x} = 0 \right\}.$$

We will use a bold font to denote vector function spaces $\mathbf{H}^1(\Omega) := (H^1(\Omega))^3$ and $\mathbf{H}_0^1(\Omega) := (H_0^1(\Omega))^3$, and other vector function spaces will be denoted analogously. The divergence-free (div-free) subspace of $\mathbf{H}^1$ will be denoted by $\mathbf{V}$:

$$\mathbf{V} := \{ \mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \},$$

and $\mathbf{V}_0 = \mathbf{V} \cap \mathbf{H}_0^1(\Omega)$.

The Poincaré inequality will be used throughout our analysis: $\forall\ \phi \in H^1(\Omega)$ satisfying $\phi = 0$ on a set of positive measure on $\partial\Omega$,

$$\|\phi\| \le C_{PF} \|\nabla \phi\|. \tag{2.1}$$

We now state a lemma for bounding the trilinear forms that arise in the analysis.

Lemma 2.1. *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ *and at least one is in* $\mathbf{H}_0^1(\Omega)$. *Then there exists a constant* M *dependent only on the size of* Ω *which satisfies*

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}, \mathbf{w}) &\le M \|\nabla \mathbf{u}\| \|\mathbf{v}\| \|\nabla \mathbf{w}\|, \\ 2(D(\mathbf{u})\mathbf{v}, \mathbf{w}) &\le \begin{cases} M \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|, \\ M \|\mathbf{u}\| \|\nabla \mathbf{v}\| (\|\mathbf{w}\|_{L^\infty} + \|\nabla \mathbf{w}\|_{L^3}). \end{cases} \end{aligned} \tag{2.2}$$

Proof. These results follow from Holder's inequality, Ladyzhenskaya inequalities, the Poincaré inequality, and the Sobolev imbedding theorems. ∎

3. The VVH formulation for equilibrium Navier–Stokes flow.

Now we consider the VVH system for equilibrium flow,

$$-\nu \Delta \mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla P = \mathbf{f}, \tag{3.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3.2}$$

$$-\nu \Delta \mathbf{w} + 2D(\mathbf{w})\mathbf{u} - \nabla hel = \mathbf{g}, \tag{3.3}$$

$$\nabla \cdot \mathbf{w} = 0, \tag{3.4}$$

where, as above, *hel* and P denote the zero-mean helical density and pressure, and the system is equipped with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \tag{3.5}$$

$$\mathbf{w}|_{\partial\Omega} = \boldsymbol{\psi}. \tag{3.6}$$

We recall that (3.1)–(3.6) is formally equivalent to the steady state Navier–Stokes system if $\mathbf{g} = \nabla \times \mathbf{f}$ and $\boldsymbol{\psi} := \nabla \times \mathbf{u}|_{\partial\Omega}$. However, in this section we assume $\mathbf{g}$ and $\boldsymbol{\psi}$ to be arbitrary from $\mathbf{H}^{-1}$ and $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ for the reasons explained below. The function $\boldsymbol{\psi}$ is assumed to satisfy the consistency condition $\int_{\partial\Omega} \boldsymbol{\psi} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ stands for an outward normal vector to $\partial\Omega$.

The following lemma proves an a priori upper bound on solutions to (3.1)–(3.6). The lemma does not exploit the relation $\mathbf{w} = \nabla \times \mathbf{u}$, which does not hold for arbitrary $\boldsymbol{\psi}$, $\mathbf{g}$, and $\mathbf{f}$. Such a treatment helps us to bridge from the differential equations to finite element formulations where, in general, $\mathbf{w}_h \neq \nabla \times \mathbf{u}_h$, and to illuminate the specifics of the discrete analysis. The bound on the vorticity gradient requires a small data condition, and we note that it is the same sufficient condition used to prove that solutions to the usual velocity-pressure equilibrium formulation are unique [15].

Lemma 3.1. *Solutions to the system* (3.1)–(3.6) *satisfy the following a priori bounds:*

$$\|\nabla \mathbf{u}\| \le \nu^{-1} \|\mathbf{f}\|_{-1}, \tag{3.7}$$

and if the data satisfies $\alpha := 1 - M\nu^{-2}\|\mathbf{f}\|_{-1} > 0$, *then*

$$\|\nabla \mathbf{w}\| < C \left(\nu^{-1}\alpha^{-1}(\|\mathbf{g}\|_{-1} + \nu^{-1}\|\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}\|\mathbf{f}\|_{-1}) + \|\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega} \right). \tag{3.8}$$

Proof. Multiply (3.1) by $\mathbf{u}$ and (3.2) by P, add the equations, and integrate over Ω. After integrating by parts, we get

$$\nu\|\nabla \mathbf{u}\|^2 = (\mathbf{f}, \mathbf{u}) \le \frac{\nu}{2}\|\nabla \mathbf{u}\|^2 + \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2,$$

which proves the velocity estimate in (3.7).

For a given $\boldsymbol{\psi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, consider its Stokes extension into Ω: $\mathbf{w}^* \in \mathbf{H}^1(\Omega)$ such that $\mathbf{w}^*|_{\partial\Omega} = \boldsymbol{\psi}$, $\nabla \cdot \mathbf{w}^* = 0$, $(\nabla \mathbf{w}^*, \nabla \mathbf{v}) = 0 \ \forall\, \mathbf{v} \in \mathbf{V}_0$. Standard a priori estimates yield

$$\|\nabla \mathbf{w}^*\| \le C\, \|\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}. \tag{3.9}$$

We let $\mathbf{w} = \overline{\mathbf{w}} + \mathbf{w}^*$, where $\overline{\mathbf{w}} \in \mathbf{H}_0^1(\Omega)$ is the weak solution to

$$-\nu\Delta\overline{\mathbf{w}} + 2\mathbf{D}(\overline{\mathbf{w}})\mathbf{u} - \nabla hel = \mathbf{g} - 2\mathbf{D}(\mathbf{w}^*)\mathbf{u} + \nu\Delta\mathbf{w}^*, \quad \nabla \cdot \overline{\mathbf{w}} = 0 \quad \text{in } \Omega. \tag{3.10}$$

For the vorticity bound in (3.8), multiply (3.10) by $\overline{\mathbf{w}}$, integrate over Ω, and then integrate by parts to get

$$\begin{aligned} \nu\|\nabla\overline{\mathbf{w}}\|^2 &= (\mathbf{g}, \overline{\mathbf{w}}) - (2D(\overline{\mathbf{w}})\mathbf{u}, \overline{\mathbf{w}}) - (2D(\mathbf{w}^*)\mathbf{u}, \overline{\mathbf{w}}) \\ &\le (\mathbf{g}, \overline{\mathbf{w}}) + M(\|\nabla\overline{\mathbf{w}}\|^2\|\nabla\mathbf{u}\| + \|\nabla\mathbf{w}^*\|\|\nabla\mathbf{u}\|\|\nabla\overline{\mathbf{w}}\|) \\ &\le \|\mathbf{g}\|_{-1}\|\nabla\overline{\mathbf{w}}\| + M(\nu^{-1}\|\nabla\overline{\mathbf{w}}\|^2\|\mathbf{f}\|_{-1} + C\,\nu^{-1}\|\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}\|\mathbf{f}\|_{-1}\|\nabla\overline{\mathbf{w}}\|). \end{aligned}$$

Simplifying by using α gives

$$\|\nabla\overline{\mathbf{w}}\| \le \alpha^{-1}(\nu^{-1}\|\mathbf{g}\|_{-1} + MC\,\nu^{-2}\|\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}\|\mathbf{f}\|_{-1}). \tag{3.11}$$

The triangle inequality $\|\nabla\mathbf{w}\| \le \|\nabla\overline{\mathbf{w}}\| + \|\nabla\mathbf{w}^*\|$, (3.11), and (3.9) yield (3.8). ∎

4. Discrete scheme. The Galerkin finite element method for the VVH formulation is as follows. Let $(\mathbf{X}_h, Q_h) \subset (\mathbf{H}^1(\Omega), L^2(\Omega))$ be conforming finite element spaces on a regular mesh τ_h of a polyhedral domain Ω, satisfying the LBB condition, inverse inequality, and the approximation properties

$$\inf_{\mathbf{v}_h \in \mathbf{X}_h} (\|\boldsymbol{\phi} - \mathbf{v}_h\|_0 + h\|\boldsymbol{\phi} - \mathbf{v}_h\|_1) \le Ch^{k+1}|\boldsymbol{\phi}|_{k+1},$$

$$\inf_{q_h \in Q_h} \|r - q_h\|_0 \le Ch^k |r|_k.$$

We denote by $\mathbf{V}_h$ the discretely div-free subspace of $\mathbf{X}_h$,

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h \ : \ (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \forall q_h \in Q_h\}.$$

Define the subspaces $\mathbf{X}_{h0} := \mathbf{X}_h \cap \mathbf{H}^1_0(\Omega)$ and $\mathbf{V}_{h0} := \mathbf{V}_h \cap \mathbf{H}^1_0(\Omega)$.

We assume homogeneous boundary conditions for velocity to simplify the weak formulation and the following analysis. The finite element formulation reads as follows: Given forcing $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and kinematic viscosity $\nu > 0$, find $(\mathbf{u}_h, \mathbf{w}_h, P_h, hel_h) \in (\mathbf{X}_{h0}, \mathbf{X}_h, Q_h, Q_h)$ satisfying, $\forall\ (\mathbf{v}_h, \boldsymbol{\chi}_h, q_h, r_h) \in (\mathbf{X}_{h0}, \mathbf{X}_{h0}, Q_h, Q_h)$,

$$\begin{cases} (\mathbf{w}_h \times \mathbf{u}_h, \mathbf{v}_h) - (P_h, \nabla \cdot \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ (\nabla \cdot \mathbf{u}_h, q_h) = 0, \\ 2(D(\mathbf{w}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) + (hel_h, \nabla \cdot \boldsymbol{\chi}_h) + \nu(\nabla \mathbf{w}_h, \nabla \boldsymbol{\chi}_h) = (\nabla \times \mathbf{f}, \boldsymbol{\chi}_h), \\ (\nabla \cdot \mathbf{w}_h, r_h) = 0 \end{cases} \tag{4.1}$$

and

$$\mathbf{w}_h = I_h(\nabla \times \mathbf{u}_h) \text{ on } \partial\Omega. \tag{4.2}$$

Here I_h denotes a generic interpolant such that $\int_{\partial\Omega} I_h(\nabla \times \mathbf{u}_h) \cdot \mathbf{n} = 0$; e.g., a Clement-type interpolation based on local averaging, or an orthogonal projector (in L^2 or H^1), or even a higher order finite difference reconstruction of boundary nodal values for vorticity [27] can be used. Specific assumptions on the definition of $\mathbf{w}_h$ on $\partial\Omega$ will be made for the purpose of stability and error analysis in sections 4.1 and 4.2.

Below, we adopt the notation that $I_h^C : \mathbf{H}^k \to \mathbf{X}_h$, $k \ge 1$, denotes a Clement-type interpolation operator satisfying

$$\|\boldsymbol{\phi} - I_h^C(\boldsymbol{\phi})\|_\ell \le Ch^{k-\ell}\|\boldsymbol{\phi}\|_k, \quad \ell = 0, \dots, k,$$

and $I_h^S : \mathbf{V} \to \mathbf{V}_h$ denotes the discretely div-free preserving interpolation operator from [13] (see also [5]).

Since $(\mathbf{X}_h, Q_h)$ is assumed to satisfy the discrete inf-sup condition, the system (4.1) can be rewritten as follows: Find $(\mathbf{u}_h, \mathbf{w}_h) \in (\mathbf{V}_{h0}, \mathbf{V}_h)$ satisfying, $\forall\ (\mathbf{v}_h, \boldsymbol{\chi}_h) \in (\mathbf{V}_{h0}, \mathbf{V}_{h0})$,

$$\begin{cases} (\mathbf{w}_h \times \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ 2(D(\mathbf{w}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) + \nu(\nabla \mathbf{w}_h, \nabla \boldsymbol{\chi}_h) = (\nabla \times \mathbf{f}, \boldsymbol{\chi}_h) \end{cases} \tag{4.3}$$

and (4.2).

4.1. Stability. In this section we prove that the velocity solutions are stable. As for the stability of the discrete vorticity, we note the following. A priori estimates of Lemma 3.1 suggest that the discrete vorticity estimate in the H^1 norm would involve the estimate of $\|I_h(\nabla \times \mathbf{u}_h)\|_{\frac{1}{2},\partial\Omega}$. By the trace theorem this calls for the estimate of $\|\nabla I_h(\nabla \times \mathbf{u}_h)\|_{L^2(\Omega)}$. This is a kind of H^2 estimate of the finite element velocity solution which we are not able to show. The way around this difficulty is to study the model (somewhat less practical) situation of "exact" vorticity boundary conditions,

$$\mathbf{w}_h = \boldsymbol{\psi}_h \text{ on } \partial\Omega \tag{4.4}$$

with some given $\boldsymbol{\psi}_h$, which is in general an approximation of $\nabla \times \mathbf{u}$ on $\partial\Omega$. We may assume $\boldsymbol{\psi}_h = I_h^C(\nabla \times \mathbf{u})|_{\partial\Omega}$ and $\int_{\partial\Omega} \boldsymbol{\psi}_h \cdot \mathbf{n} = 0$. This assumption is made exclusively for the purpose of analysis, and since it offers significantly more technical challenges than the unrealistic case of homogeneous Dirichlet vorticity boundary conditions, this analysis is a step toward the case of more practical interest.

We note that an error analysis in primitive (velocity-pressure) variables typically requires the H^{k+1} velocity solution smoothness to show the convergence of order k in the energy norm. Likewise, to prove the *vorticity* convergence of order k in the "energy" norm, the extra H^{k+2} smoothness of *velocity* solution to (1.1)–(1.4) is needed. One can try to avoid such an extra regularity assumption if the vorticity is handled in the L^2 norm. Hence we will prove certain results for the L^2 norm discrete vorticity stability and convergence. It is common that handling L^2 solution and error norms is technically based on duality arguments, which involves additional regularity assumptions. Thus, at those points we will require Ω be such that the solution to the Stokes problem is H^2-regular [7].

We now prove some technical results important to the analysis that follows. Define the operator $A_h^{-1} : \mathbf{L}^2(\Omega) \to \mathbf{V}_{h0}$ as the solution operator to the discrete Stokes problem:

$$(\nabla A_h^{-1}\boldsymbol{\psi}, \nabla \mathbf{v}_h) = (\boldsymbol{\psi}, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_{h0}. \tag{4.5}$$

LEMMA 4.1. *Assume Ω is such that the Stokes problem is H^2-regular. For any $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ it holds that*

$$\|A_h^{-1}\boldsymbol{\psi}\|_{L^\infty} + \|\nabla A_h^{-1}\boldsymbol{\psi}\|_{L^3} \le C_0\,\|\boldsymbol{\psi}\| \tag{4.6}$$

and, for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $q \in L^2(\Omega)$, and $\boldsymbol{\phi} \in \mathbf{H}^1(\Omega)$,

$$|(\mathbf{f}, \nabla \times A_h^{-1}\boldsymbol{\psi})| \le C(\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\|)\|\boldsymbol{\psi}\|, \tag{4.7}$$

$$|(q, \nabla \cdot A_h^{-1}\boldsymbol{\psi})| \le C(\|q\|_{-1} + h\|q\|)\|\boldsymbol{\psi}\|, \tag{4.8}$$

$$|(\nabla\boldsymbol{\phi}, \nabla A_h^{-1}\boldsymbol{\psi})| \le C(\|\boldsymbol{\phi}\| + \|\boldsymbol{\phi}\|_{-\frac{1}{2},\partial\Omega} + h\|\nabla\boldsymbol{\phi}\|)\|\boldsymbol{\psi}\|. \tag{4.9}$$

Proof. Denote $\mathbf{u}_h := A_h^{-1}\boldsymbol{\psi}$ and let $\mathbf{u}$ be the Stokes projection of $\boldsymbol{\psi}$, i.e., $(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\boldsymbol{\psi}, \mathbf{v})\ \forall\ \mathbf{v} \in \mathbf{V}_0$. Thanks to the H^2-regularity assumption, one gets the estimates

$$\|\mathbf{u} - \mathbf{u}_h\| \le c\,h^2\|\boldsymbol{\psi}\| \quad \text{and} \quad \|\mathbf{u}\|_2 \le c\,\|\boldsymbol{\psi}\|. \tag{4.10}$$

Using the triangle inequality, finite element inverse inequalities, the approximation property of $I_h(\mathbf{u})$, and (4.10), and embedding $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$, we estimate

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty} &\le \|\mathbf{u}_h - I_h(\mathbf{u})\|_{L^\infty} + \|I_h(\mathbf{u}) - \mathbf{u}\|_{L^\infty}\\ &\le c\,(h^{-\frac{3}{2}}\|\mathbf{u}_h - I_h(\mathbf{u})\|_{L^2} + \|\mathbf{u}\|_{L^\infty})\\ &\le c\,(h^{-\frac{3}{2}}\|\mathbf{u}_h - \mathbf{u}\|_{L^2} + h^{-\frac{3}{2}}\|\mathbf{u} - I_h(\mathbf{u})\|_{L^2} + \|\mathbf{u}\|_2)\\ &\le c\,(h^{\frac{1}{2}}\|\boldsymbol{\psi}\| + h^{\frac{1}{2}}\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2) \le c\,\|\boldsymbol{\psi}\|.\end{aligned}$$

Using this and (4.10), we get the bound for the first term on the left-hand side of (4.6):

$$\|A_h^{-1}\boldsymbol{\psi}\|_{L^\infty} = \|\mathbf{u}_h\|_{L^\infty} \le \|\mathbf{u}_h - \mathbf{u}\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \le c\,(\|\boldsymbol{\psi}\| + \|\mathbf{u}\|_2) \le C_0\,\|\boldsymbol{\psi}\|.$$

The bound for the second term on the left-hand side of (4.6) is proved by similar arguments.

To prove (4.7), denote by $A^{-1}\boldsymbol{\psi}$ the solution of the continuous counterpart for (4.5). Due to the well-known finite element error estimates for the Stokes problem and the H^2-regularity assumption, we get

$$\begin{aligned}|(\mathbf{f}, \nabla\times A_h^{-1}\boldsymbol{\psi})| &\le |(\mathbf{f}, \nabla\times A^{-1}\boldsymbol{\psi})| + |(\mathbf{f}, \nabla\times(A_h^{-1}\boldsymbol{\psi} - A^{-1}\boldsymbol{\psi})|\\ &\le \|\mathbf{f}\|_{-1}\|A^{-1}\boldsymbol{\psi}\|_2 + h\|\mathbf{f}\|\|A^{-1}\boldsymbol{\psi}\|_2\\ &\le C(\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\|)\|\boldsymbol{\psi}\|.\end{aligned}$$

The bound (4.8) is proved by the same arguments used to prove (4.7). Using further the trace inequality $\|\nabla A^{-1}\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega} \le c\,\|A^{-1}\boldsymbol{\psi}\|_2 \le c\|\boldsymbol{\psi}\|$, we obtain by similar considerations

$$\begin{aligned}|(\nabla\boldsymbol{\phi}, \nabla A_h^{-1}\boldsymbol{\psi})| &\le |(\nabla\boldsymbol{\phi}, \nabla A^{-1}\boldsymbol{\psi})| + |(\nabla\boldsymbol{\phi}, \nabla(A_h^{-1}\boldsymbol{\psi} - A^{-1}\boldsymbol{\psi})|\\ &\le \|\boldsymbol{\phi}\|\|A^{-1}\boldsymbol{\psi}\|_2 + |(\boldsymbol{\phi}, (\nabla A^{-1}\boldsymbol{\psi})\mathbf{n})_{\partial\Omega}| + h\|\nabla\boldsymbol{\phi}\|\|A^{-1}\boldsymbol{\psi}\|_2\\ &\le \|\boldsymbol{\phi}\|\|\boldsymbol{\psi}\| + \|\boldsymbol{\phi}\|_{-\frac{1}{2},\partial\Omega}\|\nabla A^{-1}\boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega} + h\|\nabla\boldsymbol{\phi}\|\|A^{-1}\boldsymbol{\psi}\|_2\\ &\le C(\|\boldsymbol{\phi}\| + \|\boldsymbol{\phi}\|_{-\frac{1}{2},\partial\Omega} + h\|\nabla\boldsymbol{\phi}\|)\|\boldsymbol{\psi}\|. \qquad \square\end{aligned}$$

We are now in a position to prove a stability result for the discrete VVH scheme.

THEOREM 4.2. *Velocity solutions to* (4.3) *are unconditionally stable; i.e., they satisfy*

$$\|\nabla\mathbf{u}_h\| \le \nu^{-1}\|\mathbf{f}\|_{-1}. \tag{4.11}$$

If the data satisfies $\alpha := 1 - M\nu^{-2}\|\mathbf{f}\|_{-1} > 0$, *then*

$$\|\nabla\mathbf{w}_h\| < C\,\left(\nu^{-1}\alpha^{-1}(\|\mathbf{f}\| + \|\boldsymbol{\psi}_h\|_{\frac{1}{2},\partial\Omega}\|\mathbf{f}\|_{-1}) + \nu^{-1}\|\boldsymbol{\psi}_h\|_{\frac{1}{2},\partial\Omega}\right) =: C_w. \tag{4.12}$$

Further, assume the H^2*-regularity condition and* $\beta := 1 - C_0\,M\nu^{-2}\|\mathbf{f}\|_{-1} > 0$. *Then vorticity solutions to* (4.3), (4.4) *satisfy*

$$\|\mathbf{w}_h\| \le c\,\beta^{-1}\left[\nu^{-1}(\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\|) + (1 + M\nu^{-2}\|\mathbf{f}\|_{-1})\|\mathbf{w}_h^*\| + \|\boldsymbol{\psi}_h\|_{-\frac{1}{2},\partial\Omega}\right], \tag{4.13}$$

where $\mathbf{w}_h^*$ *is an arbitrary function from* $\mathbf{V}_h$ *such that* $\mathbf{w}_h^*|_{\partial\Omega} = \boldsymbol{\psi}_h$.

Proof. The proof of (4.11) and (4.12) follows precisely the lines of the proof of Lemma 3.1. Thus we will prove (4.13).

Let $\mathbf{w}_h^*$ be any function from $\mathbf{V}_h$ such that $\mathbf{w}_h^*|_{\partial\Omega} = \boldsymbol{\psi}_h$, for example, $\mathbf{w}_h^* = I_h^S(\nabla\times\mathbf{u})$ if $\boldsymbol{\psi}_h = I_h^S(\nabla\times\mathbf{u})|_{\partial\Omega}$. Now the vorticity solution of (4.3) can be decomposed as

$$\mathbf{w}_h = \mathbf{w}_h^* + \overline{\mathbf{w}}_h,$$

with $\overline{\mathbf{w}}_h \in \mathbf{V}_{h0}$. From the vorticity equation in (4.1) we immediately get for any $\boldsymbol{\chi}_h \in \mathbf{V}_{h0}$

$$\nu(\nabla\overline{\mathbf{w}}_h, \nabla\boldsymbol{\chi}_h) + 2(D(\overline{\mathbf{w}}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) = (\nabla\times\mathbf{f}, \boldsymbol{\chi}_h) - \nu(\nabla\mathbf{w}_h^*, \nabla\boldsymbol{\chi}_h) - 2(D(\mathbf{w}_h^*)\mathbf{u}_h, \boldsymbol{\chi}_h).$$

We set $\boldsymbol{\chi}_h = A_h^{-1}\overline{\mathbf{w}}_h$ and further apply (2.2), the results of Lemma 4.1, Young's inequality, the inverse inequality, and (4.11) to get

$$\begin{aligned}
\nu\|\overline{\mathbf{w}}_h\|^2 &= (\nabla\times\mathbf{f}, A_h^{-1}\overline{\mathbf{w}}_h) - (2D(\overline{\mathbf{w}}_h)\mathbf{u}_h, A_h^{-1}\overline{\mathbf{w}}_h) - (2D(\mathbf{w}_h^*)\mathbf{u}_h, A_h^{-1}\overline{\mathbf{w}}_h)\\
&\quad - \nu(\nabla\mathbf{w}_h^*, \nabla A_h^{-1}\overline{\mathbf{w}}_h)\\
&\le (\mathbf{f}, \nabla\times A_h^{-1}\overline{\mathbf{w}}_h) + M(\|\overline{\mathbf{w}}_h\|\|\nabla\mathbf{u}_h\|(\|A_h^{-1}\overline{\mathbf{w}}_h\|_{L^\infty} + \|\nabla A_h^{-1}\overline{\mathbf{w}}_h\|_{L^3})\\
&\quad + \|\mathbf{w}_h^*\|\|\nabla\mathbf{u}_h\|(\|A_h^{-1}\overline{\mathbf{w}}_h\|_{L^\infty} + \|\nabla A_h^{-1}\overline{\mathbf{w}}_h\|_{L^3}))\\
&\quad + \nu C(\|\mathbf{w}_h^*\| + h\|\nabla\mathbf{w}_h^*\| + \|\boldsymbol{\psi}_h\|_{-\frac12,\partial\Omega})\|\overline{\mathbf{w}}_h\|\\
&\le C_0\,(\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\| + \nu\|\mathbf{w}_h^*\| + \nu h\|\nabla\mathbf{w}_h^*\| + \nu\|\boldsymbol{\psi}_h\|_{-\frac12,\partial\Omega})\|\overline{\mathbf{w}}_h\|\\
&\quad + C_0\,M(\|\overline{\mathbf{w}}_h\|^2\|\nabla\mathbf{u}_h\| + \|\mathbf{w}_h^*\|\|\nabla\mathbf{u}_h\|\|\overline{\mathbf{w}}_h\|)\\
&\le C_0\,(\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\| + \nu\|\mathbf{w}_h^*\| + \nu\|\boldsymbol{\psi}_h\|_{-\frac12,\partial\Omega})\|\overline{\mathbf{w}}_h\|\\
&\quad + C_0\,M(\nu^{-1}\|\overline{\mathbf{w}}_h\|^2\|\mathbf{f}\|_{-1} + \nu^{-1}\|\mathbf{f}\|_{-1}\|\mathbf{w}_h^*\|\|\overline{\mathbf{w}}_h\|).
\end{aligned}$$

Thus,

$$\begin{aligned}
(\nu - C_0\,M\nu^{-1}\|\mathbf{f}\|_{-1})\|\overline{\mathbf{w}}_h\|^2 \le C_0\,(&\|\mathbf{f}\|_{-1} + h\|\mathbf{f}\| + \nu\|\mathbf{w}_h^*\| + \nu\|\boldsymbol{\psi}_h\|_{-\frac12,\partial\Omega}\\
&+ \nu^{-1}M\|\mathbf{f}\|_{-1}\|\mathbf{w}_h^*\|)\|\overline{\mathbf{w}}_h\|,
\end{aligned}$$

and so with the small data condition we get

$$\|\overline{\mathbf{w}}_h\| \le \beta^{-1}C_0\left[\nu^{-1}\|\mathbf{f}\|_{-1} + h\nu^{-1}\|\mathbf{f}\| + (1 + M\nu^{-2}\|\mathbf{f}\|_{-1})\|\mathbf{w}_h^*\| + \|\boldsymbol{\psi}_h\|_{-\frac12,\partial\Omega}\right].$$

The vorticity estimate (4.13) now follows due to the triangle inequality. ∎

Remark 4.3. For the more practical vorticity boundary conditions $\mathbf{w}_h = I_h(\nabla\times\mathbf{u}_h)$ on $\partial\Omega$ (instead of the model condition (4.4)), consider the stability bound (4.13). Assume $\mathbf{w}_h^* = I_h(\nabla\times\mathbf{u}_h)$ is discretely div-free. Then the term $\|\mathbf{w}_h^*\|$ can be perfectly bounded through $\|\mathbf{w}_h^*\| = \|I_h(\nabla\times\mathbf{u}_h)\| \le \|\nabla\mathbf{u}_h\| \le \nu^{-1}\|\mathbf{f}\|_{-1}$. However, we are lacking a bound for $\|I_h(\nabla\times\mathbf{u}_h)\|_{-\frac12,\partial\Omega}$ by $\|\nabla\mathbf{u}_h\|$ and hence by a norm of $\mathbf{f}$. This seems to be the bottleneck of extending the analysis for the case of $\mathbf{w}_h = I_h(\nabla\times\mathbf{u}_h)$ on $\partial\Omega$.

With the help of Theorem 4.2, we can prove existence and uniqueness for the discrete VVH system.

Lemma 4.4. *Solutions to* (4.3), (4.4) *exist on a given regular mesh. If the data satisfies* $\alpha > 0$ *and* $M^2C_wC_{PF}\alpha^{-1}\nu^{-3}\|\mathbf{f}\|_{-1} < 1$, *solutions are unique.*

Proof. The bound on solutions given by Theorem 4.2 is sufficient for a Leray–Schauder-type fixed point theorem to prove solution existence [15] on a fixed regular mesh. For uniqueness, suppose there are two solutions. Denote $\mathbf{e}_u = \mathbf{u}_1 - \mathbf{u}_2$ and

$\mathbf{e}_w = \mathbf{w}_1 - \mathbf{w}_2$. Note that both $\mathbf{e}_u$ and $\mathbf{e}_w$ satisfy homogeneous Dirichlet boundary conditions. We then have from (4.3) that

$$\nu\|\nabla \mathbf{e}_u\|^2 \leq M\|\mathbf{e}_w\|\|\nabla \mathbf{u}_2\|\|\nabla \mathbf{e}_u\|, \tag{4.14}$$

$$\nu\|\nabla \mathbf{e}_w\|^2 \leq M\|\nabla \mathbf{e}_w\|^2\|\nabla \mathbf{u}_2\| + M\|\nabla \mathbf{w}_1\|\|\nabla \mathbf{e}_u\|\|\nabla \mathbf{e}_w\|. \tag{4.15}$$

Using the stability bounds of solutions from Theorem 4.2, these equations reduce to

$$\|\nabla \mathbf{e}_u\| \leq M\nu^{-2}\|\mathbf{f}\|_{-1}\|\mathbf{e}_w\|, \tag{4.16}$$

$$\nu\alpha\|\nabla \mathbf{e}_w\|^2 \leq MC_w\|\nabla \mathbf{e}_u\|\|\nabla \mathbf{e}_w\|. \tag{4.17}$$

Substituting (4.16) into (4.17), we get

$$\nu\alpha\|\nabla \mathbf{e}_w\|^2 \leq M^2 C_w C_{PF}\nu^{-2}\|\mathbf{f}\|_{-1}\|\nabla \mathbf{e}_w\|^2, \tag{4.18}$$

from which uniqueness of vorticity solutions follows from the small data assumption. Uniqueness of velocity solutions then follows from (4.16). ∎

4.2. Error estimate. In this section we derive an a priori finite element error estimate. We assume that

$$\mathbf{w}_h = I_h^C(\nabla \times \mathbf{u}) \quad \text{on} \quad \partial\Omega. \tag{4.19}$$

Due to the definition of the vorticity $\mathbf{w}$ and based on the stability result of Theorem 4.2, it would be natural to derive an error estimate for $\mathbf{u}_h$ in the H^1 norm and $\mathbf{w}_h$ in the L^2 norm. We start with a technical result which helps us to handle nonhomogeneous vorticity boundary conditions in the error analysis. For the treatment of the nonhomogeneous boundary condition, we need the following quantities defined for sufficiently smooth $\boldsymbol{\xi}$ on Ω:

$$\gamma_h^t(\boldsymbol{\xi}) := \sup_{\boldsymbol{\phi}\in\mathbf{H}^2} \|\boldsymbol{\phi}\|_2^{-1} \int_{\partial\Omega} (\boldsymbol{\xi} - I_h^C(\boldsymbol{\xi}))\cdot\frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}}, \quad \gamma_h^n(\boldsymbol{\xi}) := \sup_{\phi\in H^1} \|\phi\|_1^{-1}\int_{\partial\Omega}(\boldsymbol{\xi} - I_h^C(\boldsymbol{\xi}))\cdot\mathbf{n}\,\phi.$$

We will comment on γ_h^t and γ_h^n in Remark 4.6 after we prove the following lemma.

LEMMA 4.5. *Assume that* $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $k \geq 1$, *and* $\partial\Omega$ *is sufficiently regular such that, for the solution to*

$$-\nu\Delta\mathbf{w}^* + \nabla q = 0, \quad \nabla\cdot\mathbf{w}^* = 0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{w}^*\,|_{\partial\Omega} = (\nabla\times\mathbf{u})\,|_{\partial\Omega}, \tag{4.20}$$

it holds that $\mathbf{w}^* \in \mathbf{H}^k(\Omega)$, $q \in H^{k-1}(\Omega)$, *and* $\|\mathbf{w}^*\|_k + \|q\|_{k-1} \leq C\,\|\nabla\times\mathbf{u}\|_k$. *Further, let* $\mathbf{w}_h^*$ *be the solution of,* $\forall\,(\mathbf{v}_h, r_h) \in (\mathbf{X}_{h0}, Q_h)$,

$$\nu(\nabla\mathbf{w}_h^*, \nabla\mathbf{v}_h) - (q_h, \nabla\cdot\mathbf{v}_h) + (r_h, \nabla\cdot\mathbf{w}_h^*) = 0 \quad \text{and} \quad \mathbf{w}_h^*\,|_{\partial\Omega} = I_h^C(\nabla\times\mathbf{u})\,|_{\partial\Omega}. \tag{4.21}$$

Then we have the estimate

$$\|\mathbf{w}^* - \mathbf{w}_h^*\| \leq C_1\left(h^k + \gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u})\right). \tag{4.22}$$

Furthermore, if $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega) \cap \mathbf{H}_0^1(\Omega)$, *then*

$$\|\mathbf{w}^* - \mathbf{w}_h^*\|_1 \leq C\,h^k. \tag{4.23}$$

Proof. We decompose $\mathbf{w}^* = (\nabla \times \mathbf{u}) + \mathbf{w}^0$ and $\mathbf{w}_h^* = I_h^C(\nabla \times \mathbf{u}) + \mathbf{w}_h^0$, where $\mathbf{w}^0$ and $\mathbf{w}_h^0$ vanish on $\partial\Omega$, $\nabla \cdot \mathbf{w}^0 = 0$. Define the bilinear form on $\{\mathbf{H}_0^1, L_0^2\} \times \{\mathbf{H}_0^1, L_0^2\}$:

$$a(\mathbf{u}, p; \mathbf{v}, q) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}).$$

This bilinear form is continuous and inf-sup stable (both on $\{\mathbf{H}_0^1, L_0^2\}$ and $\{\mathbf{X}_{h0}, Q_h\}$) with respect to the norm $\|\mathbf{u}, p\| := (\|\nabla \mathbf{u}\|^2 + \|p\|^2)^{\frac{1}{2}}$. The error $\mathbf{e}^0 := \mathbf{w}^0 - \mathbf{w}_h^0$, $e := q - q_h$ satisfies for any $\mathbf{v}_h \in \mathbf{X}_{h0}$, $r_h \in Q_h$

$$a(\mathbf{e}^0, e; \mathbf{v}_h, r_h) = -\nu(\nabla(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u})), \nabla \mathbf{v}_h) + (r_h, \nabla \cdot I_h^C(\nabla \times \mathbf{u})). \tag{4.24}$$

Let $\mathbf{w}_I$ be the best possible approximation to $\mathbf{w}^0$ in $\mathbf{X}_h$ with respect to the $\|\nabla \cdot \|$ norm and let q_I be the best possible approximation to q in Q_h with respect to the L^2 norm. The identity (4.24), the inf-sup stability, and continuity of the bilinear form imply that

$$\begin{aligned}
\|\mathbf{w}_I - \mathbf{w}_h^0, q_I - q_h\| &\le C \sup_{\mathbf{v}_h, r_h \in \mathbf{X}_{h0} \times Q_h} \frac{a(\mathbf{w}_I - \mathbf{w}_h^0, q_I - q_h; \mathbf{v}_h, r_h)}{\|\mathbf{v}_h, r_h\|} \\
&\le C \sup_{\mathbf{v}_h, r_h \in \mathbf{X}_{h0} \times Q_h} \frac{a(\mathbf{w}_I - \mathbf{w}^0, q_I - q; \mathbf{v}_h, r_h)}{\|\mathbf{v}_h, r_h\|} \\
&\quad + \sup_{\mathbf{v}_h, r_h \in \mathbf{X}_{h0} \times Q_h} \frac{\nu|(\nabla(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u})), \nabla \mathbf{v}_h)| + |(r_h, \nabla \cdot I_h^C(\nabla \times \mathbf{u}))|}{\|\mathbf{v}_h, r_h\|} \\
&\le C \left(\|\mathbf{w}_I - \mathbf{w}^0, q_I - q\| + \|\nabla(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u}))\| + \|\nabla \cdot (\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u}))\| \right).
\end{aligned}$$

With the help of this estimate and the triangle inequality, we get

$$\|\mathbf{e}^0, e\| \le C \left(\inf_{\mathbf{v}_h, r_h \in \mathbf{X}_{h0} \times Q_h} \|\mathbf{w}^0 - \mathbf{v}_h, q - r_h\| + \|\nabla(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u}))\| \right).$$

From the approximation properties of our finite element spaces, we get

$$\|\mathbf{e}^0, e\| \le \begin{cases} c\, h^{k-1} & \text{if } \mathbf{u} \in \mathbf{H}^{k+1}(\Omega), \\ c\, h^k & \text{if } \mathbf{u} \in \mathbf{H}^{k+2}(\Omega). \end{cases} \tag{4.25}$$

Using the triangle inequality and approximation properties of I_h^C once again, we prove (4.23).

Now, to obtain the estimate of $\|\mathbf{w}_h^* - \mathbf{w}^*\|$ in the L^2 norm, we apply the duality argument. To this end consider $\boldsymbol{\phi} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ and $\lambda \in H^1(\Omega)$ solving

$$-\nu\, \Delta \boldsymbol{\phi} - \nabla \lambda = \mathbf{e}^0, \quad \nabla \cdot \boldsymbol{\phi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \boldsymbol{\phi}\,|_{\partial\Omega} = 0. \tag{4.26}$$

Due to the assumption on Ω, it holds that

$$\|\boldsymbol{\phi}\|_2 + \|\lambda\|_1 \le C\, \|\mathbf{e}^0\|. \tag{4.27}$$

Multiplying the equations in (4.26) by $\mathbf{e}^0$ and e, respectively, integrating by parts, and using (4.24), one gets for arbitrary $\mathbf{v}_h \in \mathbf{V}_{h0}$, $\lambda_h \in Q_h$

$$\begin{aligned}
\|\mathbf{e}^0\|^2 = \nu\, (\nabla \mathbf{e}^0, \nabla(\boldsymbol{\phi} - \mathbf{v}_h)) - (\nabla \cdot (\boldsymbol{\phi} - \mathbf{v}_h), e) + (\nabla \cdot \mathbf{e}^0, \lambda - \lambda_h) \\
- \nu(\nabla(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u})), \nabla \mathbf{v}_h) + (\nabla \cdot I_h^C(\nabla \times \mathbf{u}), \lambda_h).
\end{aligned} \tag{4.28}$$

We estimate the first three terms on the right-hand side of (4.28) in the standard manner using approximation properties and (4.27) as

$$\nu(\nabla \mathbf{e}^0, \nabla(\boldsymbol{\phi} - \mathbf{v}_h)) - (\nabla \cdot (\boldsymbol{\phi} - \mathbf{v}_h), e) + (\nabla \cdot \mathbf{e}^0, \lambda - \lambda_h) \\ \leq c\, h(\|\boldsymbol{\phi}\|_2 + \|\lambda\|_1)\, \|\mathbf{e}^0, e\| \leq c\, h \|\mathbf{e}^0, e\|\, \|\mathbf{e}^0\|.$$

The fourth and the fifth terms on the right-hand side of (4.28) are estimated separately:

$$(\nabla(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})), \nabla\mathbf{v}_h) = (\nabla(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})), \nabla(\mathbf{v}_h - \boldsymbol{\phi})) \\ - (\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}), \Delta\boldsymbol{\phi}) + \int_{\partial\Omega} (\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\cdot \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}} \\ \leq c\left(h\|\nabla(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\| + \|\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})\| + \gamma_h^t(\nabla\times\mathbf{u})\right)\|\boldsymbol{\phi}\|_2 \\ \leq c\,(h^k + \gamma_h^t(\nabla\times\mathbf{u}))\|\mathbf{e}^0\|,$$

and

$$- (\nabla\cdot I_h^C(\nabla\times\mathbf{u}), \lambda_h) = -(\nabla\cdot(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})), \lambda - \lambda_h) \\ + (\nabla\cdot(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})), \lambda) \\ \leq c\left(h\|\nabla\cdot(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\| + \|\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})\|\right)\|\lambda\|_1 \\ + \int_{\partial\Omega}(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\cdot\mathbf{n}\lambda \\ \leq c\,(h^k + \gamma_h^n(\nabla\times\mathbf{u}))\|\mathbf{e}^0\|.$$

Here we assumed only $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$. Thus with this assumption we get

$$\|\mathbf{e}^0\| \leq c\,(h\|\mathbf{e}^0, e\| + h^k + \gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u})).$$

Through (4.25) and the triangle inequality, we get (4.22). ☐

By the definition of $\mathbf{w}^*$ in (4.21), we have the standard estimate [12]

$$\|\nabla\mathbf{w}^*\| \leq C\,\|\nabla\times\mathbf{u}\|_{H^{1/2}(\Omega)} \leq C_2\|\mathbf{u}\|_2 \tag{4.29}$$

with C_2 independent of ν.

Remark 4.6. The L^2 estimate of the lemma involves $\gamma_h^t(\nabla\times\mathbf{u})$ and $\gamma_h^n(\nabla\times\mathbf{u})$. Regarding these quantities we note the following. Due to the standard extension theorem, we have the bounds

$$\gamma_h^t(\nabla\times\mathbf{u}) \leq \|(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\otimes\mathbf{n}\|_{-\frac{1}{2},\partial\Omega}, \\ \gamma_h^n(\nabla\times\mathbf{u}) \leq \|(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\cdot\mathbf{n}\|_{-\frac{1}{2},\partial\Omega}. \tag{4.30}$$

The results of Bramble and Scott [4] suggest that there exists an interpolant such that the norms on the right-hand side are $O(h^k)$ for $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$. However, we are not aware if this result holds for a typical nodal or local-averaging interpolant. If additional $\frac{1}{2}$-regularity is assumed, i.e., $\mathbf{u} \in \mathbf{H}^{k+\frac{1}{2}}(\Omega)$, one may argue that

$$\gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u}) \leq \|\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u})\|_{L^2(\partial\Omega)} \leq h^k\|\mathbf{u}\|_{k+\frac{1}{2}}.$$

Moreover, if the discretely div-free interpolant I_h^S is used instead of I_h^C, then the $\gamma_h^n(\nabla\times\mathbf{u})$ term disappears from (4.22), and, due to the $\Delta = \nabla(\nabla\cdot) - \nabla\times(\nabla\times)$ identity, the $\gamma_h^t(\boldsymbol{\xi})$ can be shown to reduce to

$$\gamma_h^t(\boldsymbol{\xi}) := \sup_{\boldsymbol{\phi}\in\mathbf{V}} \|\boldsymbol{\phi}\|_1^{-1} \int_{\partial\Omega} (\boldsymbol{\xi} - I_h(\boldsymbol{\xi}))\times\mathbf{n}\cdot\boldsymbol{\phi}$$

with the bound

$$\gamma_h^t(\nabla\times\mathbf{u}) \le \|(\nabla\times\mathbf{u} - I_h(\nabla\times\mathbf{u}))\times\mathbf{n}\|_{-\frac{1}{2},\partial\Omega}.$$

Also, in [2] it is shown that an optimal approximation order for $\gamma_h^t(\nabla\times\mathbf{u})$ can be gained if $I_h(\nabla\times\mathbf{u})|_{\partial\Omega}$ is the $L^2(\partial\Omega)$ projection of $(\nabla\times\mathbf{u})|_{\partial\Omega}$ on the space of traces of finite element functions. However, the result in [2] was proved only for linear finite elements, and we are not aware of an extension to higher order elements. Finally, with the additional assumption that $\mathbf{u}\in\mathbf{H}^{k+2}(\Omega)$, one may consider only the energy-type estimate (4.23) for which γ_h^t and γ_t^n are not relevant.

THEOREM 4.7. *Let $(\mathbf{u}, p)$ be a solution to the NSEs* (1.1)–(1.4). *Assume $\mathbf{f}\in\mathbf{L}^2(\Omega)$, $\mathbf{u}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^{k+1}(\Omega)$ for $k\ge 1$, $\mathbf{w}=\nabla\times\mathbf{u}$, and assume $\partial\Omega$ is such that the assumption of Lemma* 4.5 *holds. If $(\mathbf{u}_h, P_h)$, $(\mathbf{w}_h, hel_h)$ are the solutions to* (4.1), (4.19) *with a small data assumption on $\|\mathbf{f}\|$, we have an a priori error estimate*

$$(4.31)\quad \|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\mathbf{w}-\mathbf{w}_h\|^2$$
$$\le C\left(h^{2k} + \|(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\otimes\mathbf{n}\|^2_{-\frac{1}{2},\partial\Omega} + \|(\nabla\times\mathbf{u} - I_h^C(\nabla\times\mathbf{u}))\cdot\mathbf{n}\|^2_{-\frac{1}{2},\partial\Omega}\right).$$

Proof. Define $P = p + \frac{1}{2}\mathbf{u}^2$, $\mathbf{w}=\nabla\times\mathbf{u}$, and $hel = \mathbf{w}\cdot\mathbf{u}$; then (Theorem 2.1 in [22]) $(\mathbf{u}, P, \mathbf{w}, hel)$ solves (3.1)–(3.4) and satisfies the boundary conditions (3.5)–(3.6) with $\boldsymbol{\psi}=\nabla\times\mathbf{u}$ and $\mathbf{g}=\nabla\times\mathbf{f}$. For helical density we use embedding inequalities to show that

$$\|hel\|_k = \|\mathbf{w}\cdot\mathbf{u}\|_k \le \left(|\mathbf{w}|_k\|\mathbf{u}\|_{L^\infty} + \sum_{i=1}^k |\mathbf{w}|_{W_4^{k-i}}|\mathbf{u}|_{W_4^i}\right)^{\frac{1}{2}}$$
$$\le C\left(|\mathbf{w}|_k\|\mathbf{u}\|_{k+1} + \sum_{i=1}^k |\mathbf{w}|_{k-i+1}|\mathbf{u}|_{i+1}\right)^{\frac{1}{2}} \le C\|\mathbf{w}\|_k\|\mathbf{u}\|_{k+1}\le C.$$

Similarly one checks a bound for P. Thus it holds that $P\in H^k(\Omega)$, $\mathbf{w}\in\mathbf{H}^k(\Omega)$, $hel\in H^k(\Omega)$, and[2]

$$\|P\|_k + \|\mathbf{w}\|_k + \|hel\|_k \le C\left(\|\mathbf{u}\|_{k+1} + \|p\|_k\right) \le C.$$

Multiply (3.1) by arbitrary $\mathbf{v}_h\in\mathbf{V}_{h0}$, integrate over the domain, and subtract the first equation of (4.3). Denoting $\mathbf{e}_u = \mathbf{u}-\mathbf{u}_h$, $\mathbf{e}_w = \mathbf{w}-\mathbf{w}_h$, we have

$$(4.32)\quad \nu(\nabla\mathbf{e}_u, \nabla\mathbf{v}_h) + (\mathbf{w}\times\mathbf{e}_u, \mathbf{v}_h) + (\mathbf{e}_w\times\mathbf{u}_h, \mathbf{v}_h) - (P, \nabla\cdot\mathbf{v}_h) = 0\quad \forall\, \mathbf{v}_h\in\mathbf{V}_{h0}.$$

We next write the velocity error as pieces in and out of the finite element space $\mathbf{V}_{h0}$, as $\mathbf{e}_u = \mathbf{u}-\mathbf{u}_h = (\mathbf{u} - I_h^S(\mathbf{u})) - (\mathbf{u}_h - I_h^S(\mathbf{u})) =: \boldsymbol{\eta}_u - \boldsymbol{\phi}_h$. Choosing $\mathbf{v}_h = \boldsymbol{\phi}_h$ in (4.32),

$$(4.33)\quad \nu\|\nabla\boldsymbol{\phi}_h\|^2 = \nu(\nabla\boldsymbol{\eta}_u, \nabla\boldsymbol{\phi}_h) + (\mathbf{w}\times\boldsymbol{\eta}_u, \boldsymbol{\phi}_h) + (\mathbf{e}_w\times\mathbf{u}_h, \boldsymbol{\phi}_h) - (P, \nabla\cdot\boldsymbol{\phi}_h),$$

[2]For the proof, $hel\in\mathbf{H}^{k-1}(\Omega)$ is sufficient.

where the second and third terms on the right-hand side are bounded using (2.2), (3.8), (4.11) as

$$(\mathbf{w} \times \boldsymbol{\eta}_u, \boldsymbol{\phi}_h) \le M\|\nabla \mathbf{w}\|\|\boldsymbol{\eta}_u\|\|\nabla \boldsymbol{\phi}_h\| \le C\|\boldsymbol{\eta}_u\|\|\nabla \boldsymbol{\phi}_h\|,$$

$$(\mathbf{e}_w \times \mathbf{u}_h, \boldsymbol{\phi}_h) \le M\nu^{-1}\|\mathbf{f}\|_{-1}\|\mathbf{e}_w\|\|\nabla \boldsymbol{\phi}_h\|.$$

For arbitrary $q_h \in Q_h$, the Cauchy–Schwarz and Young's inequalities and approximation estimates with $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $P \in H^k(\Omega)$ reduce (4.33) to

$$\begin{aligned}\frac{\nu}{2}\|\nabla \boldsymbol{\phi}_h\|^2 &\le C\left(\|\nabla \boldsymbol{\eta}_u\|^2 + \|\boldsymbol{\eta}_u\|^2 + \inf_{q_h \in Q_h}\|P - q_h\|^2\right) + M\nu^{-1}\|\mathbf{f}\|_{-1}\|\mathbf{e}_w\|\|\nabla \boldsymbol{\phi}_h\| \\ &\le Ch^{2k} + M\nu^{-1}\|\mathbf{f}\|_{-1}\|\mathbf{e}_w\|\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{4.34}$$

We now turn to the vorticity equation. Define $\mathbf{w}^*$, $\mathbf{w}_h^*$ as in Lemma 4.5. Letting $\mathbf{w} = \overline{\mathbf{w}} + \mathbf{w}^*$ and $\mathbf{w}_h = \overline{\mathbf{w}}_h + \mathbf{w}_h^*$, we have from (4.3) that

$$\begin{aligned}&\nu(\nabla \overline{\mathbf{w}}_h, \nabla \boldsymbol{\chi}_h) + 2(D(\overline{\mathbf{w}}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) \\ &\quad = (\nabla \times \mathbf{f}, \boldsymbol{\chi}_h) - 2(D(\mathbf{w}_h^*)\mathbf{u}_h, \boldsymbol{\chi}_h) \quad \forall\, \boldsymbol{\chi}_h \in \mathbf{V}_{h0}\end{aligned} \tag{4.35}$$

and, from (4.20) and (3.3) with $\mathbf{g} = \nabla \times \mathbf{f}$, that

$$\begin{aligned}&\nu(\nabla \overline{\mathbf{w}}, \nabla \boldsymbol{\chi}) + 2(D(\overline{\mathbf{w}})\mathbf{u}, \boldsymbol{\chi}) + (hel + q, \nabla \cdot \boldsymbol{\chi}) \\ &\quad = (\nabla \times \mathbf{f}, \boldsymbol{\chi}) - 2(D(\mathbf{w}^*)\mathbf{u}, \boldsymbol{\chi}) \quad \forall\, \boldsymbol{\chi} \in \mathbf{H}_0^1(\Omega).\end{aligned} \tag{4.36}$$

Note that $\overline{\mathbf{w}}$ and $\overline{\mathbf{w}}_h$ satisfy the homogeneous boundary condition. Also, using (4.29),

$$\|\nabla \overline{\mathbf{w}}\| \le \|\nabla(\nabla \times \mathbf{u})\| + \|\nabla \mathbf{w}^*\| \le C\,\|\mathbf{u}\|_2 := C_3. \tag{4.37}$$

Taking $\boldsymbol{\chi} = \boldsymbol{\chi}_h$ in (4.36) and subtracting (4.35) from (4.36), we have

$$\begin{aligned}&\nu(\nabla(\overline{\mathbf{w}} - \overline{\mathbf{w}}_h), \nabla \boldsymbol{\chi}_h) + 2(D(\overline{\mathbf{w}} - \overline{\mathbf{w}}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) + 2(D(\overline{\mathbf{w}})\mathbf{e}_u, \boldsymbol{\chi}_h) + (hel + q - q_h, \nabla \cdot \boldsymbol{\chi}_h) \\ &\quad = -\,2(D(\mathbf{w}^* - \mathbf{w}_h^*)\mathbf{u}_h, \boldsymbol{\chi}_h) + 2(D(\mathbf{w}^*)\mathbf{e}_u, \boldsymbol{\chi}_h) \quad \forall\, \boldsymbol{\chi}_h \in \mathbf{V}_{h0},\ \forall\, q_h \in Q_h.\end{aligned} \tag{4.38}$$

Letting $\overline{\mathbf{w}} - \overline{\mathbf{w}}_h = \boldsymbol{\eta}_{\overline{\mathbf{w}}} - \boldsymbol{\varphi}_h$, where $\boldsymbol{\eta}_{\overline{\mathbf{w}}} := \overline{\mathbf{w}} - I_h^S(\overline{\mathbf{w}})$ and $\boldsymbol{\varphi}_h := I_h^S(\overline{\mathbf{w}}) - \overline{\mathbf{w}}_h$, (4.38) can be reduced to

$$\begin{aligned}\nu(\nabla \boldsymbol{\varphi}_h, \nabla \boldsymbol{\chi}_h) &= \nu(\nabla \boldsymbol{\eta}_{\overline{\mathbf{w}}}, \nabla \boldsymbol{\chi}_h) + 2(D(\boldsymbol{\eta}_{\overline{\mathbf{w}}})\mathbf{u}_h, \boldsymbol{\chi}_h) - 2(D(\boldsymbol{\varphi}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) \\ &\quad + 2(D(\overline{\mathbf{w}})\mathbf{e}_u, \boldsymbol{\chi}_h) + 2(D(\mathbf{w}^* - \mathbf{w}_h^*)\mathbf{u}_h, \boldsymbol{\chi}_h) - 2(D(\mathbf{w}^*)\mathbf{e}_u, \boldsymbol{\chi}_h) \\ &\quad + (hel + q - q_h, \nabla \cdot \boldsymbol{\chi}_h) \quad \forall\, \boldsymbol{\chi}_h \in \mathbf{V}_{h0},\ \forall\, q_h \in Q_h.\end{aligned} \tag{4.39}$$

Let $\boldsymbol{\chi}_h = A_h^{-1}\boldsymbol{\varphi}_h$ in (4.39), where A_h^{-1} is the operator defined in (4.5). The left-hand side of (4.39) becomes

$$\nu(\nabla \boldsymbol{\varphi}_h, \nabla A_h^{-1}\boldsymbol{\varphi}_h) = \nu\|\boldsymbol{\varphi}_h\|^2 \tag{4.40}$$

by (4.5). Using the estimates (2.2), (4.6), (4.11), (4.22), (4.29), (4.37) and Young's inequality, the trilinear terms in the right-hand side are bounded as

$$2(D(\boldsymbol{\varphi}_h)\mathbf{u}_h, A_h^{-1}\boldsymbol{\varphi}_h) \le MC_0\nu^{-1}\|\mathbf{f}\|_{-1}\|\boldsymbol{\varphi}_h\|^2, \tag{4.41}$$

$$2(D(\overline{\mathbf{w}})\mathbf{e}_h, A_h^{-1}\boldsymbol{\varphi}_h) \le C_0C_3M\|\mathbf{e}_u\|\|\boldsymbol{\varphi}_h\|, \tag{4.42}$$

$$\begin{aligned} 2(D(\boldsymbol{\eta}_{\overline{\mathbf{w}}})\mathbf{u}_h, A_h^{-1}\boldsymbol{\varphi}_h) &\le C_0M\nu^{-1}\|\mathbf{f}\|_{-1}\|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|\|\boldsymbol{\varphi}_h\| \\ &\le C\|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2 + \epsilon\|\boldsymbol{\varphi}_h\|^2, \end{aligned} \tag{4.43}$$

$$\begin{aligned} 2(D(\mathbf{w}^* - \mathbf{w}_h^*)\mathbf{u}_h, A_h^{-1}\boldsymbol{\varphi}_h) &\le C(h^k + \gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u}))\nu^{-1}\|\mathbf{f}\|_{-1}\|\boldsymbol{\varphi}_h\| \\ &\le C(h^{2k} + [\gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u})]^2) + \epsilon\|\boldsymbol{\varphi}_h\|^2, \end{aligned} \tag{4.44}$$

$$2(D(\mathbf{w}^*)\mathbf{e}_u, A_h^{-1}\boldsymbol{\varphi}_h) \le C_0C_2M\|\mathbf{u}\|_2\|\mathbf{e}_u\|\|\boldsymbol{\varphi}_h\|. \tag{4.45}$$

We apply (4.8) to estimate the last term in (4.39):

$$(hel + q - q_h, \nabla\cdot A_h^{-1}\boldsymbol{\varphi}_h) \le C(\|hel + q - q_h\|_{-1} + h\|hel + q - q_h\|)\|\boldsymbol{\varphi}_h\|.$$

Since q_h is arbitrary from Q_h, we may choose q_h, following [4], such that $\|hel + q - q_h\|_{-1} + h\|hel + q - q_h\| \le h^k(\|hel + q\|_{k-1})$. The norm $\|hel\|_{k-1}$ is bounded by the assumption of the theorem and $\|q\|_{k-1} \le c\,\|\mathbf{u}\|_{k+1} \le C$ by the definition of q in Lemma 4.5. Therefore we get

$$(hel + q - q_h, \nabla\cdot A_h^{-1}\boldsymbol{\varphi}_h) \le Ch^{2k} + \epsilon\|\boldsymbol{\varphi}_h\|^2. \tag{4.46}$$

Further, we apply (4.9) and use $\boldsymbol{\eta}_{\overline{\mathbf{w}}}|_{\partial\Omega} = 0$ to get

$$(\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}, \nabla A_h^{-1}\boldsymbol{\varphi}_h) \le C(\|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\| + h\|\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|)\|\boldsymbol{\varphi}_h\| \le C(\|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2 + h^2\|\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2) + \epsilon\|\boldsymbol{\varphi}_h\|^2;$$

using this and collecting (4.40)–(4.46), (4.39) implies that

$$\begin{aligned} (\nu - 4\epsilon)\|\boldsymbol{\varphi}_h\|^2 \le C&\left(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2 + \|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2 + h^2\|\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2\right) \\ &+ (C_0C_3M + C_0C_2M\|\mathbf{u}\|_2)\,\|\mathbf{e}_u\|\|\boldsymbol{\varphi}_h\| \\ &+ MC_0\nu^{-1}\|\mathbf{f}\|_{-1}\|\boldsymbol{\varphi}_h\|^2. \end{aligned} \tag{4.47}$$

Let $\epsilon = \frac{\nu}{8}$ in (4.47) and combine with (4.34):

$$\begin{aligned} &\frac{\nu}{2}\|\nabla\boldsymbol{\phi}_h\|^2 + \left(\frac{\nu}{2} - MC_0\nu^{-1}\|\mathbf{f}\|_{-1}\right)\|\boldsymbol{\varphi}_h\|^2 \\ &\quad\le C\left(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2 + \|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2 + h^2\|\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2\right) \\ &\qquad + M\nu^{-1}\|\mathbf{f}\|_{-1}\|\mathbf{e}_w\|\|\nabla\boldsymbol{\phi}_h\| + K\|\mathbf{e}_u\|\|\boldsymbol{\varphi}_h\|, \end{aligned} \tag{4.48}$$

where $K = C_0C_3M + C_0C_2\|\mathbf{u}\|_2M$. Since $\mathbf{e}_u = \boldsymbol{\eta}_u - \boldsymbol{\phi}_h$ and $\mathbf{e}_w = (\mathbf{w}^* - \mathbf{w}_h^*) + (\boldsymbol{\eta}_{\overline{\mathbf{w}}} - \boldsymbol{\varphi}_h)$, using Lemma 4.5, we have

$$\begin{aligned} \|\mathbf{e}_w\|\|\nabla\boldsymbol{\phi}_h\| &\le C_1h^k\|\nabla\boldsymbol{\phi}_h\| + C_1(\gamma_h^t(\nabla\times\mathbf{u}) + \gamma_h^n(\nabla\times\mathbf{u}))\|\nabla\boldsymbol{\phi}_h\| \\ &\quad + \|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|\|\nabla\boldsymbol{\phi}_h\| + \|\boldsymbol{\varphi}_h\|\|\nabla\boldsymbol{\phi}_h\|, \\ \|\mathbf{e}_u\|\|\boldsymbol{\varphi}_h\| &\le \|\boldsymbol{\eta}_u\|\|\boldsymbol{\varphi}_h\| + C_{PF}\|\nabla\boldsymbol{\phi}_h\|\|\boldsymbol{\varphi}_h\|. \end{aligned}$$

Now, by Young's inequality and the standard interpolation error estimates for $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $\mathbf{w} \in \mathbf{H}^k(\Omega)$, and $hel \in \mathbf{H}^{k-1}(\Omega)$, (4.48) is reduced to

$$\begin{aligned} &\frac{\nu}{3}\|\nabla\boldsymbol{\phi}_h\|^2 + \left(\frac{\nu}{3} - MC_0\nu^{-1}\|\mathbf{f}\|_{-1}\right)\|\boldsymbol{\varphi}_h\|^2 \\ &\le C\left(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2 + \|\nabla\boldsymbol{\eta}_u\|^2 + \|\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2 + h^2\|\nabla\boldsymbol{\eta}_{\overline{\mathbf{w}}}\|^2\right) \\ &\qquad + (M\nu^{-1}\|\mathbf{f}\|_{-1} + C_{PF}K)\|\nabla\boldsymbol{\phi}_h\|\|\boldsymbol{\varphi}_h\| \\ &\le C(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2) + \frac{1}{2}\left(M\nu^{-1}\|\mathbf{f}\|_{-1} + C_{PF}K\right)(\|\nabla\boldsymbol{\phi}_h\|^2 + \|\boldsymbol{\varphi}_h\|^2), \end{aligned}$$

from which we obtain

$$
\begin{aligned}
&\left[\frac{\nu}{3} - \frac{1}{2}\left(M\nu^{-1}\|\mathbf{f}\|_{-1} + C_{PF}K\right)\right]\|\nabla\boldsymbol{\phi}_h\|^2 \\
&\quad + \left[\frac{\nu}{3} - MC_0\nu^{-1}\|\mathbf{f}\|_{-1} - \frac{1}{2}\left(M\nu^{-1}\|\mathbf{f}\|_{-1} + C_{PF}K\right)\right]\|\boldsymbol{\varphi}_h\|^2 \\
&\leq\ C(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2).
\end{aligned} \tag{4.49}
$$

If

$$
1 - M\nu^{-2}\|\mathbf{f}\|_{-1}\left(\frac{3}{2} + 3C_0\right) - \frac{3}{2}\nu^{-1}C_{PF}K\ >\ 0, \tag{4.50}
$$

we have from (4.49) that

$$
\|\nabla\boldsymbol{\phi}_h\|^2 + \|\boldsymbol{\varphi}_h\|^2\ \leq\ C(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2). \tag{4.51}
$$

In (4.50), M, C_0, C_{PF} are constants defined in (2.2), (4.6), (2.1). These constants are independent of ν, $\mathbf{f}$, and

$$
K = C_0C_3M + C_0C_2\|\mathbf{u}\|_2M,
$$

where $C_3 \approx \|\mathbf{u}\|_2$, $C_2 \approx 1$. Assuming $\|\mathbf{u}\|_2 \leq C\nu^{-1}\|\mathbf{f}\|$, $K \approx \nu^{-1}\|\mathbf{f}\|$. Therefore, if $\|\mathbf{f}\|$ and ν^{-1} are sufficiently small to satisfy (4.50), the triangle inequality yields

$$
\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\overline{\mathbf{w}} - \overline{\mathbf{w}}_h\|^2 \leq C(h^{2k} + \gamma_h^t(\nabla\times\mathbf{u})^2 + \gamma_h^n(\nabla\times\mathbf{u})^2). \tag{4.52}
$$

The estimate (4.31) then follows from (4.22), (4.30), (4.52) and the triangle inequality. ∎

Assuming extra solution regularity, we prove the convergence in stronger norms.

THEOREM 4.8. *In addition to the assumption of Theorem* 4.7 *we assume more regularity on* $\mathbf{u}$*, i.e.,* $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{k+2}(\Omega)$ *and the small data condition of* $\mathbf{f}$*. The following a priori estimate holds:*

$$
\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\nabla(\mathbf{w}-\mathbf{w}_h)\|^2 + \|P - P_h\| + \|hel - hel_h\| \leq C\,h^{2k}. \tag{4.53}
$$

Proof. Taking $\boldsymbol{\chi}_h = \boldsymbol{\varphi}_h$ in (4.39), using (4.23), and proceeding similarly to the proof of the previous theorem, we have

$$
\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\nabla(\overline{\mathbf{w}} - \overline{\mathbf{w}}_h)\|^2\ \leq\ Ch^{2k}. \tag{4.54}
$$

Combining this estimate with (4.23) implies that

$$
\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\nabla(\mathbf{w} - \mathbf{w}_h)\|^2\ \leq\ Ch^{2k}. \tag{4.55}
$$

Errors for the pressure and helicity terms are obtained in the standard manner using the inf-sup condition. We obtain from (3.1) and (4.1) that, for $\mathbf{v}_h \in \mathbf{X}_{h,0}$,

$$
\begin{aligned}
(P_h - q_h, \nabla\cdot\mathbf{v}_h) &= (P - q_h, \nabla\cdot\mathbf{v}_h) + \nu(\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v}_h) \\
&\quad + (\mathbf{w}\times(\mathbf{u}-\mathbf{u}_h), \mathbf{v}_h) + ((\mathbf{w}-\mathbf{w}_h)\times\mathbf{u}_h, \mathbf{v}_h)
\end{aligned} \tag{4.56}
$$

for arbitrary $q_h \in Q_h$. Using the inf-sup condition, (4.56), (2.2), (3.8), and (4.11),

$$\begin{aligned}\|P_h - q_h\| &\le C \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(P_h - q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_1} \\ &\le C \sup_{\mathbf{v}_h \in \mathbf{X}_{h0}} \frac{1}{\|\mathbf{v}_h\|_1} \left[(P - q_h, \nabla \cdot \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}_h) \right. \\ &\quad \left. + (\mathbf{w} \times (\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h) + ((\mathbf{w} - \mathbf{w}_h) \times \mathbf{u}_h, \mathbf{v}_h)\right] \\ &\le C(\|P - q_h\| + \|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \|\nabla(\mathbf{w} - \mathbf{w}_h)\|). \end{aligned} \tag{4.57}$$

Thus, we get

$$\|P - P_h\| \le C \left(\inf_{q_h \in Q_h} \|P_h - q_h\| + \|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \|\nabla(\mathbf{w} - \mathbf{w}_h)\| \right) \le Ch^{2k}.$$

The estimate for helicity

$$\|hel_h - hel\| \le C \left(\inf_{q_h \in Q_h} \|hel - q_h\| + \|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \|\nabla(\mathbf{w} - \mathbf{w}_h)\| \right) \tag{4.58}$$

is also obtained by the same argument; therefore (4.53) follows from (4.55), (4.57), (4.58), and the interpolation error bound. ◻

Remark 4.9. Commonly, velocity-vorticity formulations utilize the vorticity equation

$$\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + (\nabla \mathbf{w})\mathbf{u} - (\nabla \mathbf{u})\mathbf{w} = \nabla \times \mathbf{f} \tag{4.59}$$

supplemented with velocity Poisson problem

$$-\Delta \mathbf{u} = \mathbf{w}. \tag{4.60}$$

We do not see a straightforward argument to show the energy estimate for the velocity solution of the *finite element* counterpart of (4.59)–(4.60) (even assuming homogeneous boundary conditions for both $\mathbf{u}$ and $\mathbf{w}$). This is in contrast to the case when (1.9) is used instead of (4.60). If (1.9) is used, then our analysis is extended to the case of vorticity equation (4.59) with minor changes; e.g., the H_0^1 discrete projection is used instead of the Stokes projection.

5. Numerical experiments.

Here we provide two numerical examples, the first to confirm the predicted convergence rates, and the second to show the effectiveness of the method on a benchmark problem.

5.1. Convergence rate verification.

We now compute convergence rates of the discrete scheme to the chosen solution

$$\mathbf{u} = \langle \cos(z), \sin(z), \sin(x) \rangle, \quad p = \sin(x + y),$$

testing both boundary conditions studied herein. We use the domain $(-1, 1)^3$, $\nu = 1$, and (P_2, P_1) Taylor–Hood elements. Note that the helicity for the true solution is nontrivial: $H = -8$. The equations were solved via a splitting iteration, with initial guess $\mathbf{u} = \mathbf{w} = \mathbf{0}$. The meshes used were uniform, with the coarsest being $3 \times 3 \times 3$ and the finest $33 \times 33 \times 33$ for velocity nodes. For the velocity boundary condition, we use the interpolant of the true solution on the boundary. For the vorticity boundary

TABLE 1
Velocity, vorticity, pressure, and helical density errors and convergence rates for successive mesh refinements, using the interpolant of the curl of the continuous velocity solution at the boundary for the vorticity boundary condition.

h	$\dim(\mathbf{X}_h)$	$\dim(Q_h)$	$\|\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$	Rate
1	375	27	2.089E-2	-	1.415E-1	-
$\frac{1}{2}$	2,187	125	2.666E-3	2.97	3.571E-2	1.99
$\frac{1}{4}$	14.739	729	3.2455E-4	3.04	8.755E-3	2.03
$\frac{1}{6}$	46,875	2,197	9.567E-5	3.01	3.869E-3	2.01
$\frac{1}{8}$	107,811	4,913	4.0289E-5	3.01	2.172E-3	2.01

h	$\|\mathbf{w}-\mathbf{w}_h\|$	Rate	$\|\nabla(\mathbf{w}-\mathbf{w}_h)\|$	Rate	$\|hel-hel_h\|$	Rate	$\|p-p_h\|$	Rate
1	1.858E-2	-	1.2202E-1	-	1.115E-1	-	8.986E-2	-
$\frac{1}{2}$	2.301E-3	3.01	3.050E-2	2.00	2.745E-2	2.02	2.548E-2	1.82
$\frac{1}{4}$	2.795E-4	3.04	7.494E-3	2.02	6.658E-3	2.04	6.034E-3	2.08
$\frac{1}{6}$	8.226E-5	3.02	3.316E-3	2.01	2.936E-3	2.02	2.649E-3	2.03
$\frac{1}{8}$	3.462E-5	3.01	1.862E-3	2.01	1.647E-3	2.01	1.484E-3	2.01

TABLE 2
Velocity, vorticity, pressure, and helical density errors and convergence rates for successive mesh refinements, using the interpolant of the curl of the discrete velocity solution at the boundary for the vorticity boundary condition.

h	$\dim(\mathbf{X}_h)$	$\dim(Q_h)$	$\|\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$	Rate
1	375	27	2.090E-2	-	1.416E-1	-
$\frac{1}{2}$	2,187	125	2.676E-3	2.97	3.571E-2	1.99
$\frac{1}{4}$	14.739	729	3.330E-4	3.01	8.763E-3	2.03
$\frac{1}{6}$	46,875	2,197	1.011E-4	2.94	3.873E-3	2.01
$\frac{1}{8}$	107,811	4,913	4.423E-5	2.87	2.174E-3	2.01

h	$\|\mathbf{w}-\mathbf{w}_h\|$	Rate	$\|\nabla(\mathbf{w}-\mathbf{w}_h)\|$	Rate	$\|hel-hel_h\|$	Rate	$\|p-p_h\|$	Rate
1	1.247E-1	-	8.721E-1	-	1.124E-0	-	9.061E-2	-
$\frac{1}{2}$	2.796E-2	2.16	3.193E-1	1.45	3.075E-1	1.870	2.569E-2	1.82
$\frac{1}{4}$	6.417E-3	2.12	1.129E-1	1.50	8.397E-2	1.873	6.087E-3	2.08
$\frac{1}{6}$	2.744E-3	2.10	6.118E-2	1.51	3.930E-2	1.873	2.677E-3	2.03
$\frac{1}{8}$	1.509E-3	2.08	3.957E-2	1.51	2.289E-2	1.879	1.500E-3	2.01

condition, we compute first with the interpolant of the true vorticity (results in Table 1), and then with the (more practical) boundary value of the $\mathbf{V}_h$ projected curl of the discrete velocity (results in Table 2).

For the "interpolant of the true vorticity" boundary condition, we see from Table 1 that the rates agree with Theorems 4.7 and 4.8. Moreover, the errors $\|\mathbf{w}-\mathbf{w}_h\|$ and $\|\mathbf{u}-\mathbf{u}_h\|$ appear to have an L^2 lift in this idealized setting.

For the more practical "interpolant of the discrete vorticity" boundary condition, we computed the same convergence rates as for the idealized problem. In this setting, it is expected (and true) that all errors will increase, since the boundary condition for vorticity uses a continuous approximation of a discontinuous function already with error. First, we note that Theorem 4.7 appears to still hold for this case, since the data indicates that

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\| + \|\mathbf{w}-\mathbf{w}_h\| \approx O(h^2).$$

However, the L^2 lift for $\|\mathbf{w}-\mathbf{w}_h\|$ is no longer present; both of these error norms now converge as $O(h^2)$. The $\|\nabla(\mathbf{w}-\mathbf{w}_h)\|$ norm's convergence also deteriorates; it loses $\frac{1}{2}$ power in convergence rate from the idealized setting. The convergence of the pressure appears unaffected by the change in the boundary condition. The helical density errors are worse than for the ideal boundary condition, although from Table 2 it is unclear if the rate of convergence will tend to 2 or has less than optimal convergence.

5.2. Three-dimensional (3D) flow over a forward-backward step. We next test the scheme on the $Re = 20$ 3D flow over the forward and backward facing step problem studied by John in [14], which has as its domain a channel modeled by a $[0, 10] \times [0, 40] \times [0, 10]$ rectangular box, with a $10 \times 1 \times 1$ step on the bottom ($z = 0$) of the channel beginning 5 units into the channel. No-slip boundary conditions are prescribed on the top, bottom, and sides of the box, as well as on the step. For the inflow velocity condition, we use the $Re = 20$ steady channel flow profile, which varies from the constant inflow profile of $\mathbf{u}_{in} = \langle 0, 1, 0\rangle$ used by John; our choice is more physically plausible, and, moreover, the constant inflow profile would cause a blowup of vorticity at the inflow edges as $h \to 0$. For the outflow velocity, we enforce outflow = inflow, which is reasonable for sufficiently long channels such as the one used here. Where no-slip conditions are enforced for velocity, we impose a no-penetration condition for the vorticity. For the remaining vorticity boundary conditions, we enforce vorticity on the boundary to be the L^2 projection of the curl of the velocity into $\mathbf{V}_h$. We solve this problem using (P_2, P_1) Taylor–Hood elements and a mesh that gives $\dim(\mathbf{X}_h) = 27,693$ and $\dim(Q_h) = 1,350$, using grad-div stabilization [21] (with parameter $\gamma = 1$) in both the velocity and vorticity equations. Since the velocity equation is in rotational form, wall bounded flows can have a large (Bernoulli) pressure error that can adversely affect the velocity error; grad-div stabilization is known to reduce this effect [23, 16], and its use here provided a modest improvement in accuracy of the velocity field. A similar phenomenon is true in the vorticity equation with the helical density; grad-div stabilization provided a significant improvement in both the solution and the sensitivity of the nonlinear solver.

The solution is plotted in Figures 1–3. Figure 1 shows a speed contour plot of the sliceplane $x = 5$ with overlying streamlines for the entire channel, and, as expected, a smooth flow is observed and recirculation is seen behind the step. To see the recirculation behind the step, in Figure 2 we see the zoomed-in picture at the step with contour sliceplanes of the speed and with streamtubes showing the three-dimensionality of the recirculation. Contour slices and isosurfaces of helical density are shown near the step in Figure 3, and it is interesting to note that the larger values of helicity are outside the recirculation region; helicity is the product of velocity and vorticity, and even though vorticity is strong in the recirculation region, velocity is weaker there.

6. Conclusions and future directions. We have presented a rigorous analysis of a finite element method for the VVH formulation of the equilibrium NSEs, including stability, existence/uniqueness, and convergence, assuming an "exact" vorticity boundary condition. To the best of our knowledge, this is the first rigorous analysis of any velocity-vorticity method for approximating NSE solutions. The analysis overcomes several technical difficulties not present in the unrealistic cases of periodic

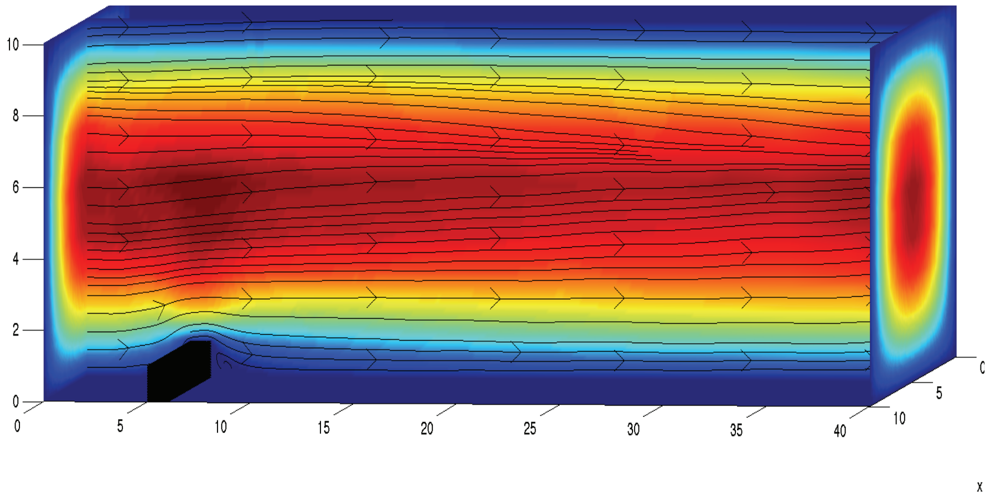

FIG. 1. *Shown above are speed contours and streamlines on the $x = 5$ sliceplane for the 3D step, and the $y = 0$ and $y = 40$ inflow and outflow speed contours.*

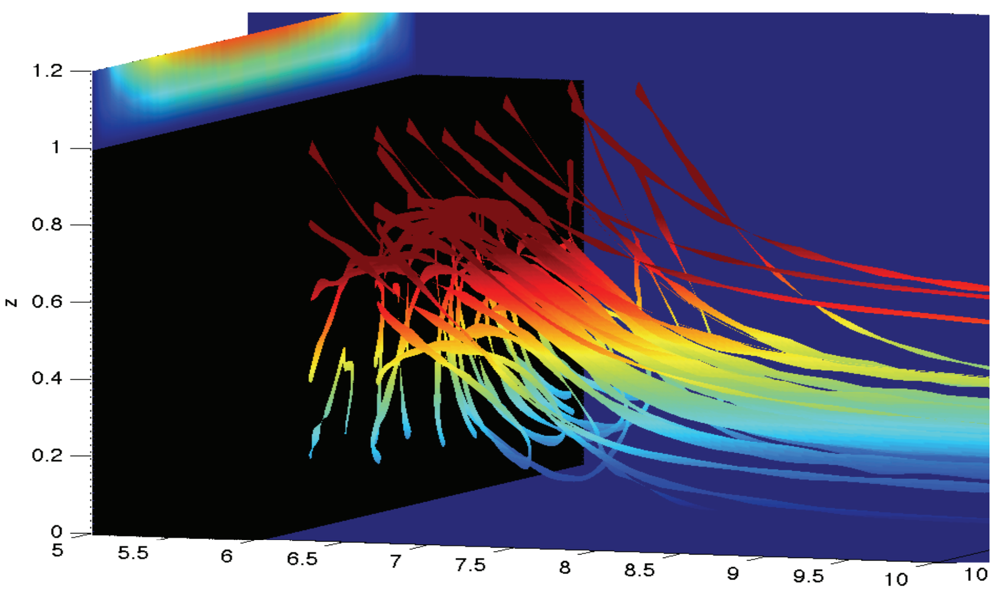

FIG. 2. *Shown above is a zoomed-in view of Figure 1 at the step, with streamtubes in the flow field displaying the three-dimensionality of the recirculation.*

or homogeneous Dirichlet vorticity boundary conditions. It is therefore an important step toward the ultimate goal of analyzing the case of "interpolant of the curl of the discrete velocity" vorticity boundary conditions, as well as the time-dependent case. In Remark 4.3 we outlined what seems to be the main technical difficulty in completing this goal within the framework of this paper. Other outstanding questions include the following: (i) Can we find an appropriate weak formulation for vorticity equations with nonhomogeneous boundary conditions (1.8)? This may or may not involve

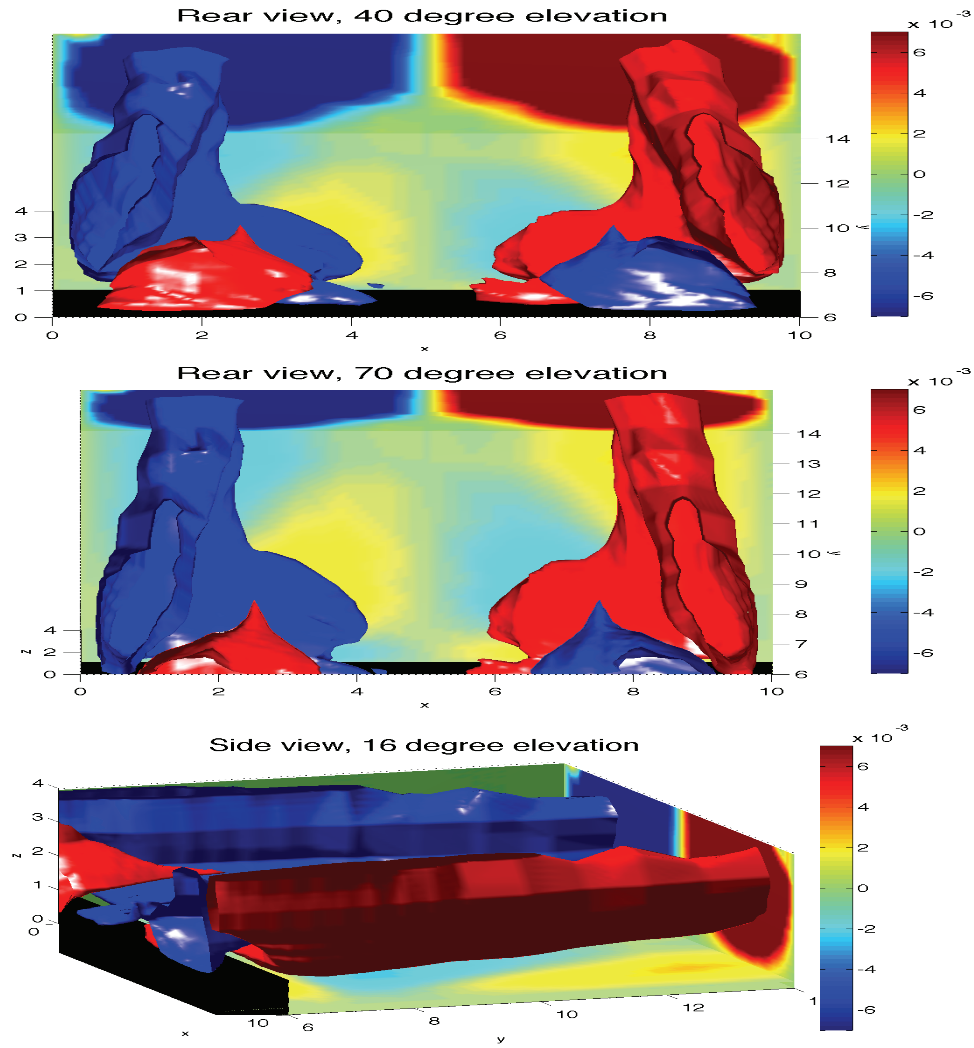

FIG. 3. *Shown above is the helicity density near the step, as isosurfaces (red for helicity density* 0.025 *and blue for* -0.025 *(color online only)) and contour slices. The step is shown in black, and three viewing angles are provided: (top) from the rear of the the step, looking down from a* 40 *degree elevation; (middle) from the rear of the step, looking down from a* 70 *degree elevation; and (bottom) from the side, looking down from a* 16 *degree elevation.*

finding appropriate functional space for $(\nabla \times \mathbf{u})|_{\partial\Omega}$ or imposing vorticity boundary conditions in a weak form. (ii) Can an estimate on the L^2 norm of the discrete vorticity be proved without a smallness data assumption? (iii) Can the energy-type bound (4.11) be proved if the (discretization of the) Poisson equation $-\Delta \mathbf{u} = \mathbf{w}$ is used instead of (1.9)?

Computational tests confirmed the theoretical results and suggested that they may still hold in the case of more realistic boundary conditions. Furthermore, we provided a benchmark test of the method for steady flow over a 3D forward and backward facing step, which gave good results.

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