Orthonormal Wavelets Arising From HDAFs

Jiansheng Yang, Lixin Shen, Manos Papadakis, Ioannis Kakadiaris, Donald J. Kouri, and David K. Hoffmam

Abstract. Hermite Distributed Approximating Functionals (HDAFs) were introduced by Hoffman and Kouri for the numerical computation of the solutions of certain types of PDEs. These functions and their Fourier transforms are both dominated by Gaussians. In this paper we modify HDAFs in order to construct a family of continuous orthonormal wavelets. These wavelets can have any prescribed number of vanishing moments. The main part of the paper is devoted to the study of the smoothness of these wavelets.

1. Introduction and Main Results

The theory of Distributed Approximating Functionals (DAF's) was introduced in the work of Hoffman, Kouri, and their collaborators (e.g. see [**HNSK91**], [**HK93**]). DAFs are infinitely differential functions very localized in the spatial domain, which, depending on a certain parameter, form sequences of functions converging to the Dirac's delta function in the sense of distributions. This property made DAFs useful for the computation of numerical solutions of certain types of PDEs. In an effort to understand their efficiency for computational applications, the rigorous study of the mathematical foundations of DAFs was intitiated in [**CG99**] and continued in [**WWH01**]. The first DAFs that were created were the Hermite DAFs (HDAFs, see [**HNSK91**]) of whom their definition we include in the next paragraph.

The present paper is primarily devoted to the study of the smoothness of the Modified Hermite DAF orthonormal scaling functions and of their corresponding orthonormal wavelets, which were introduced in [**KPS2, KPS1**]. A Hermite DAF (HDAF) of order $N = 1, 2, 3, \ldots,$ is defined by the following equation:

$$h_{N,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \sum_{n=0}^{N} \left(-\frac{1}{4}\right)^n \frac{1}{n!} H_{2n}\left(\frac{x}{\sqrt{2}\sigma}\right),$$

1991 *Mathematics Subject Classification.* Primary 42C40; Secondary 65T60.

Key words and phrases. Hermite Distributed Approximating Functionals, wavelet.

This research was supported in part by the following grants: Texas Higher Eduaction Coordinating board ARP 003652-0071-2001, NSF DMS-0070376, NSF CHE-0074311, and R.A. Welch Foundation E-0608. Additional financial support was also provided by the Texas Learning and Computation Center of the University of Houston.

where σ is a positive constant and H_{2n} is the Hermite polynomial of degree $2n$. The Hermite polynomial H_n can be expressed by Rodrigues' Formula $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ (see also [**AS72**]). The Fourier transform of the HDAF is routinely shown to be

$$\widehat{h}_{N,\sigma}(\xi) = e^{-\frac{\xi^2\sigma^2}{2}} \sum_{n=0}^{N} \frac{(\xi^2\sigma^2)^n}{2^n n!}, \tag{1.1}$$

where the Fourier transform of an L^1-function f is defined by

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}.$$

It is clear that $\lim_{N\to\infty} \widehat{h}_{N,\sigma}(\xi) = 1$ and $\lim_{\sigma\to 0} \widehat{h}_{N,\sigma}(\xi) = 1$ for all $\xi \in \mathbb{R}$. These two properties justify why HDAFs form sequences of functions converging to the Dirac's delta function in the sense of distributions (see [**CG99, WWH01**]). We will refrain from repeating the definition of multiresolution analysis (MRA), since it is widely known. We only point out that we use the definition of an MRA given in [**HW96**] and that an *orthonormal scaling function* is a square-integrable function ϕ satisfying, first, a two-scale relation, i.e.

$$\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi) \quad \text{a.e. in } \mathbb{R}, \tag{1.2}$$

where m_0 is 2π-periodic and belongs to $L^2([-\pi, \pi))$; second, the integer translates of ϕ form an orthonormal system. The closed linear span of this system is the core subspace V_0 of the MRA $\{V_j\}_j, j \in \mathbb{Z}$. The 2π-periodic function m_0 associated with ϕ is called the *low pass filter* associated with ϕ. The wavelet ψ corresponding to ϕ is given by

$$\widehat{\psi}(\xi) = e^{-i\xi/2}\overline{m_0(\xi/2 + \pi)}\widehat{\phi}(\xi/2).$$

The most widely known and utilized wavelets are these with compact support in the time domain, because the wavelet transforms induced by them are implemented by finite impulse response (FIR) filter banks. However, wavelets associated with low pass filters, which are not trigonometric polynomials are implemented by infinite impulse response (IIR) filter banks, and such filterbanks are less popular than their FIR counterparts. However, IIR filters with sharp frequency selectivity may be more desirable for certain types of applications, especially if they are symmetric as well. The low pass filters associated with the scaling functions we study in this paper have both these two properties. Next, we present the construction of the MHDAF orthonormal scaling functions. Let N be a non-negative integer and $\sigma > 0$. We define $m_{N,\sigma}$, which we also extend 2π-periodically over the real line, by the following equation:

$$m_{N,\sigma}(\xi) := \begin{cases} \sqrt{1 - \widehat{h}^2_{N,\sigma}(\xi+\pi)}, & -\pi \le \xi < -\frac{\pi}{2}, \\ \widehat{h}_{N,\sigma}(\xi), & -\frac{\pi}{2} \le \xi \le \frac{\pi}{2}, \\ \sqrt{1 - \widehat{h}^2_{N,\sigma}(\xi-\pi)}, & \frac{\pi}{2} < \xi < \pi. \end{cases}$$

We also impose the following condition:

$$\widehat{h}_{N,\sigma}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}. \tag{1.3}$$

The definition of $m_{N,\sigma}$ and (1.3) readily imply

$$|m_{N,\sigma}(\xi)|^2 + |m_{N,\sigma}(\xi+\pi)|^2 = 1, \quad \xi \in \mathbb{R}. \tag{1.4}$$

Since, $\widehat{h}_{N,\sigma}(0) = 1$ we obviously have $m_{N,\sigma}(0) = 1$ and $m_{N,\sigma}(\pm\pi) = 0$. It is also easy to see that $m_{N,\sigma}$ is a 2π-periodic continuous function.

On the other hand, $\widehat{h}_{N,\sigma}$ is decreasing in the interval $[0, +\infty)$. Therefore

$$\inf_{|\xi| \leq \pi/2} |m_{N,\sigma}(\xi)| \geq m_{N,\sigma}(\pi/2) = \frac{\sqrt{2}}{2} > 0.$$

If we combine this fact with (1.4), we obtain that ϕ, defined by

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m_{N,\sigma}(\xi/2^j), \quad \xi \in \mathbb{R}\ ,$$

is an orthonormal scaling function (see [**Mal89**] or Corollary 4.14 in [**HW96**]), which we call a *Modified-HDAF (MHDAF) scaling function.* Apparently $m_{N,\sigma}$ is the low pass filter corresponding to ϕ. We call $m_{N,\sigma}$ the *Modified-HDAF low pass filter of order N.* Since every $m_{N,\sigma}$ is even and takes on non-negative values only, the Fourier transforms of MHDAF scaling functions are even and non-negative. Thus the MHDAF scaling functions are themselves even in the time domain.

MHDAF low pass filters have an infinite number of nonzero Fourier coefficients, and the magnitude of the k-th coefficient is majorized by $O(|k|^{-3})$ ([**KPS2, KPS1**]). Thus MHDAF scaling functions are not compactly supported. This poses some technical constraints in the study of their smoothness properties, since it leaves us with one choice only: the study of the decay properties of their Fourier transforms. There is a considerable literature devoted to the study of this subject (e.g. [**CR95, HE95, V94**]). The reader may also find a in [**S99**] a brief and rather comprehensive account on the various smoothness measures of refinable tempered distributions based on the decay of their Fourier transform along with a generalization of Proposition 3.5 of [**CR95**] (Theorem 1.2). However, the conditions guaranteeing a certain degree of smoothness for scaling or refinable functions are not always easy to check, as the reader will soon realize (see also [**FS01**]). Remarkably enough, Theorem 1.4 reveals that, the Fourier domain techniques proposed in [**CR95**] for the study of the smoothness of compactly supported scaling functions (primarily Propositions 3.3 and 3.5) can also be applied in the case of the MHDAF scaling functions.

The investigation of the smoothness of the MHDAF scaling functions begins with the study of the relationship between σ and N, the two parameters that determine an HDAF. Recall that (1.3) implicitly determines σ in terms of N, which is a non-negative integer. The unique σ satisfying (1.3) will be denoted by $\sigma(N)$. The estimation of the smoothness of the MHDAF scaling function is given by Theorem 1.5. However, before obtaining this result we need several intermediate results, namely Theorems 1.1, 1.2 and 1.4. We prove these theorems in the following sections.

THEOREM 1.1. *Set $\gamma_N := \frac{\pi^2\sigma^2(N)}{8}$. Then the following are true:*

(1) *$\gamma_N < 1.08$ for $N = 1$, and $\gamma_N < N$ for $N \geq 2$;*
(2) *$\gamma_N > \frac{N+2}{3}$ for all $N \geq 1$;*

In order to improve the readability of the paper we introduce the following notation; $\sigma(N)$, $\widehat{h}_{N,\sigma(N)}$ and $m_{N,\sigma(N)}$ will be denoted by σ, $\widehat{h}_N$ and m_N, respectively.

THEOREM 1.2. *The following are true:*

(1) *For every N, the low pass filter m_N has a zero of order of $N+1$ at each $\xi = (2k+1)\pi$, with $k \in \mathbb{Z}$.*
(2) *If N is odd, then m_N is C^∞ at $(2k+1)\pi$, for all $k \in \mathbb{Z}$; if N is even, then m_N is C^N at $(2k+1)\pi$, for all $k \in \mathbb{Z}$.*
(3) *m_N is C^1 at every $k\pi + \pi/2$, $k \in \mathbb{Z}$. At all other points m_N is C^∞.*

For $\alpha = n+\beta$, where n is a non-negative integer and $0 < \beta \leq 1$, we define C^α to be the set of all bounded functions f which are n times continuously differentiable and their n-th order derivative $f^{(n)}$ is Hölder continuous of order β, i.e.

$$|f^{(n)}(x+h) - f^{(n)}(x)| \leq C|h|^\beta$$

for every $x, h \in \mathbb{R}$.

LEMMA 1.3. *([**HW96**]) Let $f \in \mathbb{L}^2(\mathbb{R})$ and $\alpha > 0$. Suppose*

$$\int_{2^j\pi}^{2^{j+1}\pi} |\hat{f}(\xi)| d\xi \leq C2^{-\alpha j} \ , \quad j = 0, 1, 2, \ldots .$$

Then the following are true: If α is not an integer, then $f \in C^\alpha$, otherwise $f \in C^{\alpha-\epsilon}$, for every $0 < \epsilon < \alpha$.

The previous technical result is Lemma 3.22 in [**HW96**]. Using Lemma 1.3 we obtain that if $|\hat{f}(\xi)| \leq C(1+|\xi|)^{-1-\alpha}$, then f belongs to C^α, if α is not an integer; otherwise $f \in C^{\alpha-\epsilon}$, for every ϵ in $(0, \alpha)$.

For each $N \geq 0$ we define the function

$$L_N(\xi) := \frac{m_N(\xi)}{|\cos^{N+1}(\xi/2)|}.$$

Theorem 1.2 implies that L_N is well-defined everywhere on $\mathbb{R}$. The following theorem is the key result we need in order to prove Theorem 1.5.

THEOREM 1.4. *The following are true:*

(1) *$L_N(\xi)$ is an increasing function on the interval $[0, \pi]$;*
(2) *$|L_N(\xi)L_N(2\xi)|$ is a decreasing function on the interval $[\frac{2\pi}{3}, \pi]$.*

¿From the previous theorem, we have

$$\begin{array}{ll} |L_N(\xi)| \leq |L_N(\frac{2\pi}{3})| & \text{for} \quad |\xi| \leq \frac{2\pi}{3}, \\ |L_N(\xi)L_N(2\xi)| \leq |L_N(\frac{2\pi}{3})|^2 & \text{for} \quad \frac{2\pi}{3} \leq |\xi| \leq \pi. \end{array}$$

This allows us to apply Propositions 3.3 and 3.5 in [**CR95**] in order to derive Theorem 1.5 below, which is the main result of this paper. Although these two results from [**CR95**] require that L_N is C^∞, it is enough to use the C^1 continuity of L_N to prove them.

THEOREM 1.5. *For every $N = 1, 2, \ldots$ we have*

$$0 \leq \widehat{\phi}(\xi) \leq C(1+|\xi|)^{-N-1+b_N}$$

with $b_N := \log|L_N(2\pi/3)|/\log 2$. Therefore, $\phi \in C^{N-b_N-\epsilon}(\mathbb{R})$, for every $\epsilon > 0$. Furthemore, $\lim_{N\to+\infty}(N - b_N) = +\infty$.

Now, from $\lim_{N\to+\infty}(N - b_N) = +\infty$, *we infer that, the smoothness of the Modified HDAF scaling functions increases, as N increases.*

2. Estimating $\sigma(N)$

In this section, we prove Theorem 1.1. We begin with a lemma, which will be frequently used in Sections 2 and 4. The proof of the lemma is straightforward, so it will be omitted.

LEMMA 2.1. *Let a, b and c be arbitrary real numbers such that $(a+c-b)(a+c) > 0$ and $bc \geq 0$. Then*

$$\frac{a-b}{a+c-b} \leq \frac{a}{a+c}. \tag{2.1}$$

PROOF OF THEOREM 1.1. Define

$$g_N(\gamma) := e^{-\gamma} \sum_{n=0}^{N} \frac{\gamma^n}{n!} = \frac{\sum_{n=0}^{N} \frac{\gamma^n}{n!}}{\sum_{n=0}^{+\infty} \frac{\gamma^n}{n!}}, \quad \gamma \in \mathbb{R}.$$

Then, $g_N'(\gamma) = -e^{-\gamma} \frac{\gamma^N}{N!} \leq 0$, so g_N is decreasing on $[0, +\infty)$. On the other hand, $g_N(\gamma_N) = \frac{\sqrt{2}}{2}$. Therefore, in order to prove the first part of Theorem 1.1, it is enough to show $g_1(1.08) < \frac{\sqrt{2}}{2}$ and $g_N(N) < \frac{\sqrt{2}}{2}$ for every $N \geq 2$. Likewise, in order to establish the second part of Theorem 1.1, it is enough to show $g_N\left(\frac{N+2}{3}\right) > \frac{\sqrt{2}}{2}$, for all $N \geq 1$.

(1) By direct computations one can verify that $g_1(1.08)$, and $g_N(N)$ for $N = 2, 3, 4$ are less than $\frac{\sqrt{2}}{2}$, hence $\gamma_1 < 1.08$ and $\gamma_N < N$ for $N = 2, 3, 4$.

Let $N \geq 5$. Now, set

$$I_1 := \sum_{n=0}^{N} \frac{N^n}{n!} = \left\{ \sum_{k=1}^{N-1} \left(\frac{N-1}{N} \cdots \frac{N-k}{N} \right) + 2 \right\} \frac{N^{N-1}}{(N-1)!} \tag{2.2}$$

and

$$I_2 := \sum_{n=N+1}^{\infty} \frac{N^n}{n!} = \left\{ \sum_{k=1}^{\infty} \left(\frac{N}{N+1} \cdots \frac{N}{N+k} \right) \right\} \frac{N^{N-1}}{(N-1)!}.$$

On the other hand, it is easy to see

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{N}{N+1} \cdots \frac{N}{N+k} \right) &> \sum_{k=1}^{N-1} \left(\frac{N}{N+1} \cdots \frac{N}{N+k} \right) \\ &> \sum_{k=1}^{N-1} \left(\frac{N-1}{N} \cdots \frac{N-k}{N} \right). \end{aligned}$$

Moreover, setting $a = N$, $b = N - 5$ and $c = n$, where $n = 1, 2, \ldots, k$, Lemma 2.1 implies

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{N}{N+1} \cdots \frac{N}{N+k} \right) &> \sum_{k=1}^{\infty} \left(\frac{5}{5+1} \cdots \frac{5}{5+k} \right) \\ &> \sum_{k=1}^{4} \left(\frac{5}{5+1} \cdots \frac{5}{5+k} \right) > 2\,. \end{aligned}$$

Combining the last two inequalities with the definitions of I_1 and I_2, we obtain

$$I_1 < 2I_2.$$

Now, using lemma 2.1 for $a = I_1 + (2I_2 - I_1)$, $b = 2I_2 - I_1$ and $c = I_2$ we derive

$$g_N(N) = \frac{I_1}{I_1 + I_2} \leq \frac{I_1 + (2I_2 - I_1)}{I_1 + I_2 + (2I_2 - I_1)} = \frac{2}{3} < \frac{\sqrt{2}}{2}.$$

The final conclusion comes from the fact that γ_N is decreasing on the positive half-axis.

(2) For all $N \geq 1$ define

$$II_1 := \sum_{n=0}^{N} \frac{(\frac{N+2}{3})^n}{n!} \quad \text{and} \quad II_2 := \sum_{n=N+1}^{\infty} \frac{(\frac{N+2}{3})^n}{n!}.$$

Then

$$II_1 \geq \frac{(\frac{N+2}{3})^{N+1}}{(N+1)!} \left(\frac{N+1}{\frac{N+2}{3}} + \frac{N+1}{\frac{N+2}{3}} \times \frac{N}{\frac{N+2}{3}} \right) \geq 4 \frac{(\frac{N+2}{3})^{N+1}}{(N+1)!}$$

and

$$II_2 \leq \frac{(\frac{N+2}{3})^{N+1}}{(N+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{3}{2} \frac{(\frac{N+2}{3})^{N+1}}{(N+1)!}.$$

Applying Lemma 2.1 again, with $a = II_1$, $b = II_1 - 4\frac{(\frac{N+2}{3})^{N+1}}{(N+1)!}$ and $c = II_2$ we obtain,

$$g_N\left(\frac{N+2}{3}\right) = \frac{II_1}{II_1 + II_2} > \frac{4}{4 + \frac{3}{2}} > \frac{\sqrt{2}}{2},$$

which implies $\gamma_N > \frac{N+2}{3}$.

□

3. The Vanishing Moments of the MHDAF Orthonormal Wavelets

In this section we prove Theorem 1.2.

PROOF OF THEOREM 1.2. By the symmetry and the periodicity of $m_N(\xi)$, we only have to check the continuity and the order of the zero of m_N at $\xi = \pi$.

Using (1.4) and $m_N(\xi + \pi) = m_N(\xi - \pi)$, which is true for all ξ, we have:

$$m_N(\xi) = \sqrt{1 - \widehat{h}_N^2(\xi - \pi)}$$

in $(\pi - \delta, \pi + \delta)$, where $0 < \delta < \frac{\pi}{2}$. For notational convenience, let us define

$$d(\xi) := \sqrt{1 - \widehat{h}_N^2(\xi)}.$$

The conclusion of Theorem 1.2 will follow, once we have established the properties of d at the origin.

Since,

$$\begin{aligned} 1 - \widehat{h}_N^2(\xi) &= (1 + \widehat{h}_N(\xi))(1 - \widehat{h}_N(\xi)) \\ &= (1 + \widehat{h}_N(\xi)) e^{-\frac{\xi^2\sigma^2}{2}} \sum_{n=N+1}^{\infty} \frac{(\xi^2\sigma^2)^n}{2^n n!} \\ &= (1 + \widehat{h}_N(\xi)) \frac{\sigma^{2(N+1)}\xi^{2(N+1)} e^{-\frac{\xi^2\sigma^2}{2}}}{2^{N+1}} \sum_{n=N+1}^{\infty} \frac{(\xi^2\sigma^2)^{n-N-1}}{2^{n-N-1} n!}, \end{aligned}$$

$d(\xi)$ can be factored as

$$d(\xi) = |\xi|^{N+1} g(\xi), \tag{3.1}$$

where

$$g(\xi) = \sqrt{(1+\widehat{h}_N(\xi)) \frac{\sigma^{2(N+1)} e^{-\frac{\xi^2\sigma^2}{2}}}{2^{N+1}} \sum_{n=N+1}^{\infty} \frac{(\xi^2\sigma^2)^{n-N-1}}{2^{n-N-1} n!}}$$

Clearly, $g(\xi) \in C^\infty$. Hence, if N is odd, then $d(\xi) \in C^\infty$, and the order of the zero of m_N at $\xi = \pi$ is equal to $N+1$.

If N is even, then d is only C^N at 0, but the order of the zero of m_N at $\xi = \pi$ is still equal to $N+1$. We ommit the proof of the last item in Theorem 1.2, because it is straightforward. □

4. The Smoothness of M-HDAF Scaling Function

This section begins with the proof of the first part of Theorem 1.4. Then, we give some technical lemmas, useful only in the proof the second part of Theorem 1.4. The proofs of the second part of Theorem 1.4 and of the last assertion of Theorem 1.5 conclude the present section.

Let us first introduce some auxilliary functions.

Let $\gamma > 0$. Define,

$$\begin{aligned}
\mathcal{Q}_{N,1}(\gamma) &:= \left(\frac{\gamma^{N+1}}{(N+1)!}\right)^{-1} \sum_{n=N+1}^{\infty} \frac{\gamma^n}{n!}, \\
\mathcal{Q}_{N,2}(\gamma) &:= \left(\frac{\gamma^N}{N!}\right)^{-1} \sum_{n=0}^{N} \frac{\gamma^n}{n!}, \\
\mathcal{Q}_{N,3}(\gamma) &:= \left(\sum_{n=0}^{N} \frac{\gamma^n}{n!}\right)^{-1} \sum_{n=N+1}^{\infty} \frac{\gamma^n}{n!}, \\
\mathcal{Q}_{N,4}(\gamma) &:= \frac{1}{\mathcal{Q}_{N,1}(\gamma)} - \frac{2}{(2+\mathcal{Q}_{N,3}(\gamma))\mathcal{Q}_{N,1}(\gamma)}.
\end{aligned}$$

Clearly,

$$\mathcal{Q}_{N,1}(\gamma) = 1 + \sum_{n=1}^{\infty} \frac{\gamma^n}{(N+2)\cdots(N+n+1)}; \tag{4.1}$$

$$\mathcal{Q}_{N,2}(\gamma) = \sum_{n=0}^{N} \frac{N!}{(N-n)!\gamma^n}; \tag{4.2}$$

$$\mathcal{Q}_{N,3}(\gamma) = \frac{\gamma}{N+1} \cdot \frac{\mathcal{Q}_{N,1}(\gamma)}{\mathcal{Q}_{N,2}(\gamma)}. \tag{4.3}$$

Lemma 4.1. *The following are true for every $N \geq 1$:*

(1) $1 - \frac{1}{\mathcal{Q}_{N,1}(\gamma)} \leq \frac{\gamma}{N+2}$;
(2) $\mathcal{Q}_{N,2}$ *is decreasing on* $[0,\infty)$, *and* $\mathcal{Q}_{N,2}(N) > \mathcal{Q}_{N-1,2}(N-1)$.
(3) $\mathcal{Q}_{N,4}(\gamma) \leq \frac{\mathcal{Q}_{N,3}(\gamma)}{2}$, *and* $\mathcal{Q}_{N,4}$ *is increasing on* $(0, N+1]$.

Proof.

(1) Using (4.1), we have

$$1-\frac{1}{\mathcal{Q}_{N,1}(\gamma)}=\frac{\frac{\gamma}{N+2}(1+\sum_{n=1}^{\infty}\frac{\gamma^n}{(N+3)\cdots(N+2+n)})}{1+\sum_{n=1}^{\infty}\frac{\gamma^n}{(N+2)\cdots(N+1+n)}}\leq\frac{\gamma}{N+2}.$$

(2) Since

$$\mathcal{Q}'_{N,2}(\gamma)=\frac{\gamma^{N-1}}{(N-1)!}\left(\sum_{n=1}^{N}\frac{\gamma^n}{(n-1)!}\left(\frac{1}{N}-\frac{1}{n}\right)-1\right)\frac{1}{\left(\frac{\gamma^N}{N!}\right)^2}<0,$$

we conclude that $\mathcal{Q}_{N,2}$ is decreasing on $[0,\infty)$. But,

$$\frac{N^N}{N!}=\frac{N^{N-1}}{(N-1)!}>\frac{(N-1)^{N-2}}{(N-2)!}$$

and

$$\frac{N-1}{N}\cdots\frac{N-k}{N}>\frac{(N-1)-1}{N-1}\cdots\frac{(N-1)-k}{N-1}$$

for every $1\leq k\leq N-1$. Then, using (2.2), we have

$$\mathcal{Q}_{N,2}(N)>\mathcal{Q}_{N-1,2}(N-1).$$

(3) Eq. (4.1) implies $\mathcal{Q}_{N,1}(\gamma)\geq 1$. This inequality and $2+\mathcal{Q}_{N,3}(\gamma)\geq 2$, for all $\gamma\geq 0$, imply

$$\mathcal{Q}_{N,4}(\gamma)=\frac{1}{\mathcal{Q}_{N,1}(\gamma)}\cdot\frac{\mathcal{Q}_{N,3}(\gamma)}{2+\mathcal{Q}_{N,3}(\gamma)}\leq\frac{\mathcal{Q}_{N,3}(\gamma)}{2}.$$

On the other hand, it is easy to check

$$\mathcal{Q}_{N,4}(\gamma)=\frac{\frac{\gamma^{N+1}}{(N+1)!}}{e^{\gamma}+\sum_{n=0}^{N}\frac{\gamma^n}{n!}}.$$

Then, we clearly have,

$$\mathcal{Q}'_{N,4}(\gamma)=\frac{\frac{\gamma^N}{N!}(e^{\gamma}+\sum_{n=0}^{N}\frac{\gamma^n}{n!})-\frac{\gamma^{N+1}}{(N+1)!}(e^{\gamma}+\sum_{n=0}^{N-1}\frac{\gamma^n}{n!})}{(e^{\gamma}+\sum_{n=0}^{N}\frac{\gamma^n}{n!})^2}>0$$

for every $\gamma\in(0,N+1]$, which implies that $\mathcal{Q}_{N,4}$ is an increasing function on the interval $(0,N+1]$.

□

Proof of item 1 of Theorem 1.4. In order to establish that L_N is increasing on the interval $[0,\pi]$, it is sufficient to prove $L'_N\geq 0$ on each one of the intervals $[0,\pi/2]$ and $[\pi/2,\pi]$.

First let $\xi\in[0,\pi/2]$. Then

$$L_N(\xi)=\frac{\widehat{h}_N(\xi)}{\cos^{N+1}(\xi/2)}.$$

In order to show $L'_N(\xi)\geq 0$, for every $\xi\in[0,\pi/2]$, it is enough to prove

$$\frac{\left(\widehat{h}_N(\xi)\right)'}{\widehat{h}_N(\xi)}\geq\frac{\left(\cos^{N+1}(\xi/2)\right)'}{\cos^{N+1}(\xi/2)},$$

equivalently,

$$\frac{1}{2}\tan(\xi/2) \geq \frac{\sigma^2\xi}{N+1}(\mathcal{Q}_{N,2}(\sigma^2\xi^2/2))^{-1}.$$

First, let $N \leq 4$. Then by item 1 of Theorem 1.1, we have

$$\frac{\sigma^2\xi^2}{2} \leq \frac{\sigma^2\pi^2}{8} = \max(1.08, N),$$

so routine calculations imply

$$\frac{\sigma^2\xi}{N+1}\left(\mathcal{Q}_{N,2}\left(\frac{\sigma^2\xi^2}{2}\right)\right)^{-1} \leq \frac{\xi}{4} \leq \frac{1}{2}\tan(\frac{\xi}{2}),$$

for $N = 1, 2, 3, 4$.

On the other hand, if $N \geq 5$, then item 1 of Theorem 1.1, gives $\sigma^2 \leq \frac{8N}{\pi^2}$. Then,

$$\frac{\sigma^2}{N+1} < \frac{8}{\pi^2}$$

and item 2 of the previous lemma implies

$$\left(\mathcal{Q}_{N,2}\left(\frac{\sigma^2\xi^2}{2}\right)\right)^{-1} \leq (\mathcal{Q}_{N,2}(N))^{-1} \leq (\mathcal{Q}_{N-1,2}(N-1))^{-1} \leq \cdots \leq (\mathcal{Q}_{5,2}(5))^{-1} < \frac{2}{7}.$$

Hence

$$\frac{\sigma^2\xi}{N+1}\left(\mathcal{Q}_{N,2}\left(\frac{\sigma^2\xi^2}{2}\right)\right)^{-1} \leq \frac{16}{7\pi^2}\xi \leq \frac{2.3\xi}{\pi^2} \leq \frac{\xi}{4} \leq \frac{1}{2}\tan(\xi/2)\ ,$$

which establishes that, for $N \geq 5$, L_N is increasing on $[0, \pi/2]$.

Now, let $\xi \in [\frac{\pi}{2}, \pi]$. Define

$$\tilde{L}_N(\xi) := L_N(\pi - \xi) = \frac{\sqrt{1-\widehat{h}_N^2(\xi)}}{\cos^{N+1}(\frac{\pi-\xi}{2})} = \frac{\sqrt{1-\widehat{h}_N^2(\xi)}}{\sin^{N+1}(\xi/2)}.$$

We must now prove that $\tilde{L}_N$ is decreasing on $[0, \frac{\pi}{2}]$. To accomplish this goal it is enough to show

$$\frac{\left(\sqrt{1-\widehat{h}_N^2(\xi)}\right)'}{\sqrt{1-\widehat{h}_N^2(\xi)}} \leq \frac{(\sin^{N+1}(\xi/2))'}{\sin^{N+1}(\xi/2)}$$

or equivalently,

$$\frac{1}{\tan\xi/2} \geq \frac{1}{\frac{\xi}{2}\mathcal{Q}_{N,1}(\frac{\sigma^2\xi^2}{2})}. \tag{4.4}$$

But, (4.1) implies $\mathcal{Q}_{N,1}(\frac{\sigma^2\xi^2}{2}) \geq 1 + \frac{\frac{\sigma^2\xi^2}{2}}{N+2}$, so (4.4) is true if the following inequality is true as well:

$$\tan(\xi/2) \leq \frac{\xi}{2} + \frac{\frac{\sigma^2\xi^3}{4}}{N+2}. \tag{4.5}$$

Now, item 2 of Theorem 1.1 implies $\sigma^2 > \frac{8}{\pi^2}\cdot\frac{N+2}{3}$. Therefore, to obtain (4.5), it is enough to prove

$$\tan\xi \leq \xi + \frac{16\xi^3}{3\pi^2}, \tag{4.6}$$

for every $\xi \in [0, \pi/4]$. So, let

$$g(\xi) := \tan\xi - \xi - \frac{16\xi^3}{3\pi^2}.$$

Then

$$\begin{aligned} g'(\xi) &= \sec^2\xi - 1 - \frac{16\xi^2}{\pi^2} \\ &= \tan^2\xi - \frac{16\xi^2}{\pi^2} \\ &= \xi^2\left(\frac{\tan^2\xi}{\xi^2} - \frac{16}{\pi^2}\right) \\ &\le \xi^2\left(\frac{\tan^2\xi}{\xi^2} - \frac{16}{\pi^2}\right)|_{\xi=\frac{\pi}{4}} = 0. \end{aligned}$$

so, $g(\xi) \le g(0) = 0$, for every $\xi \in [0, \pi/4]$. This completes the proof of (4.5) and so of the first part of Theorem 1.4. □

The proof of second part of Theorem 1.4 is more complicated and it requires several intermediate results, which we will state and prove in the sequel.

LEMMA 4.2. *Let $x \in (0, \pi/4]$. Then, the following are true:*

(1) $\frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} - \frac{1}{2\tan\frac{x}{2}} \ge \frac{x^3}{135}$;
(2) $\frac{1}{x} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} \ge \frac{x}{13}$.

PROOF. Since the proof of item 2 is easy, we will only give the proof of item 1. We first have to prove

$$\tan x \ge x + \frac{1}{3!}x^3 + \frac{16}{5!}x^5\ , \quad 0 < x \le \pi/8\ . \tag{4.7}$$

In order to prove the latter inequality we have to use Taylor's expansion of the tangent function ([**AS72**]):

$$\tan x = \sum_{n=1}^{\infty}\frac{(-4)^n(1-4^n)B_{2n}}{(2n)!}x^{2n-1}\ ,$$

where B_n is a Bernoulli number. Since,

$$B_{2n} = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{p=1}^{\infty}p^{-2n} \quad \text{for} \qquad n = 1, 2, \ldots.$$

we conclude

$$\frac{(-4)^n(1-4^n)B_{2n}}{(2n)!} > 0 \qquad n = 1, 2, \ldots.$$

This establishes (4.7). Using (4.7) we obtain

$$\begin{aligned} \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} - \frac{1}{2\tan\frac{x}{2}} &\ge \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3+\frac{16}{5!}(\frac{x}{2})^5)} \\ &\ge \frac{\frac{x^3}{120}}{(1+\frac{2}{3!}(\frac{\pi}{8})^2)(1+\frac{2}{3!}(\frac{\pi}{8})^2+\frac{16}{5!}(\frac{\pi}{8})^4)} \\ &\ge \frac{x^3}{135}\ . \end{aligned}$$

$\square$

PROPOSITION 4.3. For every $N \geq 1$ and $x \in (0, \frac{\pi}{4}]$ we have

$$\frac{1}{\frac{x}{2}(2+\mathcal{Q}_{N,3}(\frac{\sigma^2 x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2 x^2}{2})} - \frac{1}{2\tan\frac{x}{2}} + \tan x - \frac{(\frac{4\sigma^2}{N+1})x}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)} \geq 0. \tag{4.8}$$

PROOF. First, we discuss the case $N \leq 3$. Applying item 1 of Theorem 1.1 we obtain

$$\frac{\sigma^2 x^2}{2} \leq \frac{\gamma_N}{4}, \quad \text{for every} \quad x \in [0, \pi/4],$$

where $\gamma_N = \frac{\pi^2\sigma^2}{8}$. Eqs. (4.1) and (4.2) imply

$$\mathcal{Q}_{N,1}\left(\frac{\sigma^2 x^2}{2}\right) \leq \sum_{n=0}^{\infty}\left(\frac{\gamma_N}{4(N+2)}\right)^n, \quad \mathcal{Q}_{N,2}\left(\frac{\sigma^2 x^2}{2}\right) \geq \sum_{n=0}^{N}\frac{N!}{(N-n)!(\frac{\gamma_N}{4})^n}.$$

Now, let $N = 1$. According to Theorem 1.1 $\gamma_1 < 1.08$, so $\sigma^2 \leq \frac{8\times 1.08}{\pi^2}$. Therefore,

$$\mathcal{Q}_{1,1}\left(\frac{\sigma^2 x^2}{2}\right) \leq \frac{10}{9}, \quad \mathcal{Q}_{1,2}\left(\frac{\sigma^2 x^2}{2}\right) \geq \frac{127}{27}.$$

Now, applying (4.3) we get

$$\mathcal{Q}_{1,3}\left(\frac{\sigma^2 x^2}{2}\right) \leq \frac{16.2}{127}\cdot\frac{4x^2}{\pi^2}. \tag{4.9}$$

Furthermore, item 1 of Lemma 4.1 gives us

$$1 - \frac{1}{\mathcal{Q}_{1,1}(\frac{\sigma^2 x^2}{2})} \leq \frac{\sigma^2 x^2}{6} \leq \frac{4.32}{3\pi^2}x^2. \tag{4.10}$$

Using the definition of $\mathcal{Q}_{1,4}$ one can easily verify

$$\begin{aligned} &\frac{1}{\frac{x}{2}(2+\mathcal{Q}_{1,3}(\frac{\sigma^2 x^2}{2}))\mathcal{Q}_{1,1}(\frac{\sigma^2 x^2}{2})} - \frac{1}{2\tan\frac{x}{2}} \\ &= \left(\frac{1}{x} - \frac{1}{2\tan\frac{x}{2}}\right) - \left(\frac{1}{x} - \frac{1}{x\mathcal{Q}_{1,1}(\frac{\sigma^2 x^2}{2})}\right) - \frac{\mathcal{Q}_{1,4}(\frac{\sigma^2 x^2}{2})}{x}. \end{aligned} \tag{4.11}$$

Using the Taylor series expansion of the tangent and the fact that for $0 \leq x < \pi/2$ we can easily see $\tan\frac{x}{2} \geq \frac{x}{2} + \frac{2}{3!}(\frac{x}{2})^3$. Combining the previous inequality with item 2 of Lemma 4.2 we conclude

$$\frac{1}{x} - \frac{1}{2\tan\frac{x}{2}} \geq \frac{x}{13}. \tag{4.12}$$

On the other hand, (4.10) implies

$$\frac{1}{x} - \frac{1}{x\mathcal{Q}_{1,1}(\frac{\sigma^2 x^2}{2})} \leq \frac{4.32x}{3\pi^2}. \tag{4.13}$$

Item 3 of Lemma 4.1 and (4.9) finally imply

$$\frac{\mathcal{Q}_{1,4}(\frac{\sigma^2 x^2}{2})}{x} \leq \frac{8.1}{127}\cdot\frac{4x}{\pi^2}. \tag{4.14}$$

Combining inequalities (4.12), (4.13), (4.14) and (4.11) we obtain

$$\begin{aligned}\frac{1}{\frac{x}{2}(2+\mathcal{Q}_{1,3}(\frac{\sigma^2x^2}{2}))\mathcal{Q}_{1,1}(\frac{\sigma^2x^2}{2})}-\frac{1}{2\tan\frac{x}{2}} &\geq \frac{x}{13}-\frac{1.08}{3}\cdot\frac{4x}{\pi^2}-\frac{8.1}{127}\cdot\frac{4x}{\pi^2}\\ &\geq \frac{x}{13}-(0.4238)(\frac{4x}{\pi^2}).\end{aligned}$$

Hence, the left-hand-side of (4.8), which we denote by LHS, satisfies

$$LHS \quad\geq\quad \tan x+\frac{x}{13}-\left(0.4238+\frac{4.32}{\mathcal{Q}_{1,2}(2\sigma^2x^2)}\right)\frac{4x}{\pi^2},$$

so (4.8) will be valid, if the right-hand-side of previous inequality is non-negative for every $x\in(0,\frac{\pi}{4}]$. Let us first, assume $x\in(0,\frac{\pi}{6}]$. Since $\sigma^2\leq\frac{8\times1.08}{\pi^2}$ and $\mathcal{Q}_{1,2}$ is decreasing we obtain $\mathcal{Q}_{1,2}(2\sigma^2x^2)\geq\mathcal{Q}_{1,2}(\frac{4.32}{9})>3$. Therefore,

$$\tan x+\frac{x}{13}-\left(0.4238+\frac{4.32}{\mathcal{Q}_{1,2}(2\sigma^2x^2)}\right)\frac{4x}{\pi^2}\geq\tan x+\frac{x}{13}-\left(0.4238+\frac{4.32}{3}\right)\frac{4x}{\pi^2}\geq x-\frac{8x}{\pi^2}\geq 0;$$

Now, let $x\in[\frac{\pi}{6},\frac{\pi}{4}]$. Arguing as in the previous case we obtain $\mathcal{Q}_{1,2}(2\sigma^2x^2)\geq\mathcal{Q}_{1,2}(1.08)=\frac{52}{27}$. Combining these inequalities with $\tan x+\frac{x}{13}\geq\frac{7x}{6}$, we conclude

$$\tan x+\frac{x}{13}-\left(0.4238+\frac{4.32}{\mathcal{Q}_{1,2}(2\sigma^2x^2)}\right)\frac{4x}{\pi^2}\geq\frac{7x}{6}-\left(0.4238+\frac{4.32}{52/27}\right)\frac{4x}{\pi^2}\geq 0.$$

The cases $N=2$ and $N=3$ are similar to the case $N=1$, so we will ommit their proofs.

Now, let $N\geq4$. The item 1 of Theorem 1.1 implies

$$\frac{\sigma^2x^2}{2}\leq\frac{\gamma_N}{4}\leq\frac{N}{4}, \tag{4.15}$$

for every $x\in[0,\pi/4]$, where $\gamma_N=\frac{\pi^2\sigma^2}{8}$. Then, using the previous inequalities and (4.1) once again, we get

$$\mathcal{Q}_{N,1}\left(\frac{\sigma^2x^2}{2}\right)\leq\sum_{n=0}^{\infty}\left(\frac{N}{4(N+2)}\right)^n\leq\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n=\frac{4}{3}.$$

Applying the definition of $\mathcal{Q}_{N,2}$ and using (4.15), we have

$$\mathcal{Q}_{N,2}\left(\frac{\sigma^2x^2}{2}\right)\geq\left(\frac{(\frac{\sigma^2x^2}{2})^N}{N!}\right)^{-1}\sum_{n=0}^{N-1}\frac{(\frac{\sigma^2x^2}{2})^n}{n!}\geq\frac{N}{\frac{\sigma^2x^2}{2}}\left(1+\sum_{n=1}^{N-1}\frac{(N-1)\cdots(N-n)}{(\frac{N}{4})^n}\right).$$

Now, observe

$$\sum_{n=1}^{N-1}\frac{(N-1)\cdots(N-n)}{(\frac{N}{4})^n}>\sum_{n=1}^{3}\frac{(N-1)\cdots(N-n)}{(\frac{N}{4})^n}.$$

Letting $n=1,2,3$ and a,b,c in Lemma 2.1 be equal to $N-n$, $N-4$, n, respectively we obtain $(N-n)/N>(4-n)/4$. So

$$\frac{(N-1)\cdots(N-n)}{(\frac{N}{4})^n}>(4-1)\cdots(4-n),\quad n=1,2,3.$$

Therefore,

$$\mathcal{Q}_{N,2}\left(\frac{\sigma^2x^2}{2}\right)\geq\frac{N}{\frac{\sigma^2x^2}{2}}(1+\sum_{n=1}^{3}(4-1)\cdots(4-n))\geq\frac{16N}{\frac{\sigma^2x^2}{2}}.$$

Thus, using also (4.3), we conclude

$$\mathcal{Q}_{N,3}\left(\frac{\sigma^2x^2}{2}\right) \leq \frac{\sigma^2}{N+1}\cdot\frac{\sigma^2}{N}\cdot\frac{x^4}{48} \leq (\frac{8}{\pi^2})^2(\frac{x^4}{48}) \leq \frac{x^4}{72}. \tag{4.16}$$

Next, we will establish the following two inequalities, which clearly imply (4.8).

$$\frac{1}{\frac{x}{2}(2+\mathcal{Q}_{N,3}(\frac{\sigma^2x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{2\tan\frac{x}{2}} \geq \frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} \tag{4.17}$$

and

$$\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} + \tan x - \frac{(\frac{4\sigma^2}{N+1})x}{\mathcal{Q}_{N,2}(2\sigma^2x^2)} \geq 0. \tag{4.18}$$

We begin with the proof of (4.17). The definition of $\mathcal{Q}_{N,4}$ readily implies

$$\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{\frac{x}{2}(2+\mathcal{Q}_{N,3}(\frac{\sigma^2x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} = \frac{\mathcal{Q}_{N,4}(\frac{\sigma^2x^2}{2})}{x}.$$

But,

$$\frac{\mathcal{Q}_{N,4}(\frac{\sigma^2x^2}{2})}{x} \leq \frac{\mathcal{Q}_{N,3}(\frac{\sigma^2x^2}{2})}{2x},$$

due to item 3 of Lemma 4.1. Combining the previous inequality with (4.16) we get

$$\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{\frac{x}{2}(2+\mathcal{Q}_{N,3}(\frac{\sigma^2x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} \leq \frac{x^3}{144}, \tag{4.19}$$

which together with item 1 of Lemma 4.2 imply (4.17).

Let us now prove (4.18). Item 1 of lemma 4.1 and item 2 of 4.2 imply

$$\begin{aligned}\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} &= \left(\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{x}\right) + \left(\frac{1}{x} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)}\right)\\ &\geq \frac{x}{13} - \frac{\sigma^2 x}{2(N+2)} \geq \frac{x}{13} - \frac{N}{N+2}\cdot\frac{4x}{\pi^2}.\end{aligned}$$

Using the fact that $\gamma_N < N$ and the previous inequality, we obtain

$$\begin{aligned}&\frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2x^2}{2})} - \frac{1}{2(\frac{x}{2}+\frac{2}{3!}(\frac{x}{2})^3)} + \tan x - \frac{(\frac{4\sigma^2}{N+1})x}{\mathcal{Q}_{N,2}(2\sigma^2x^2)}\\ \geq\ &\tan x + \frac{x}{13} - \left(\frac{N}{N+2} + \frac{\frac{8N}{N+1}}{\mathcal{Q}_{N,2}(2\sigma^2x^2)}\right)\frac{4x}{\pi^2},\end{aligned}$$

for all $0 \leq x \leq \pi/4$. So, the conclusion of proposition 4.3 will be established once we have that the RHS of previous inequality is non-negative, for all $0 \leq x \leq \pi/4$ and $N \geq 4$.

In order to prove this fact, let us first assume $0 \leq x \leq \frac{\pi}{6}$. Using the fact that $\mathcal{Q}_{N,2}$ is decreasing we derive

$$\mathcal{Q}_{N,2}(2\sigma^2x^2) \geq \mathcal{Q}_{N,2}\left(\frac{4N}{9}\right) \geq \sum_{n=0}^{4}\frac{4!}{(4-n)!(\frac{16}{9})^n} > 8.$$

Thus,

$$\tan x + \frac{x}{13} - \left(\frac{N}{N+2} + \frac{\frac{8N}{N+1}}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)} \right) \frac{4x}{\pi^2} \geq x - \frac{8x}{\pi^2} \geq 0 \, .$$

Now, let $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$. Since, $\tan x + \frac{x}{13} \geq \frac{7x}{6}$, in order to prove

$$\tan x + \frac{x}{13} - \left(\frac{N}{N+2} + \frac{\frac{8N}{N+1}}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)} \right) \frac{4x}{\pi^2} \geq 0 \, , \tag{4.20}$$

it is enough to establish

$$\left(\frac{N}{N+2} + \frac{\frac{8N}{N+1}}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)} \right) \frac{4x}{\pi^2} \leq \frac{7x}{6} \tag{4.21}$$

for all $N \geq 4$ and $x \in (\frac{\pi}{6}, \frac{\pi}{4}]$.

Since for every $x \in (\frac{\pi}{6}, \frac{\pi}{4}]$ we have $2\sigma^2 x^2 \leq N$, the fact that $\mathcal{Q}_{N,2}$ is decreasing, implies $\mathcal{Q}_{N,2}(2\sigma^2 x^2) \geq \mathcal{Q}_{N,2}(N)$. Direct calculations show that for $N = 4, 5, 6, 7, 8, 9$, $\mathcal{Q}_{N,2}(N)$ is greater than 3.2, 3.5, 3.77, 4, 4.24, and 4.45 respectively. This establishes (4.21) for $N = 4, 5, 6, 7, 8, 9$ and $x \in (\frac{\pi}{6}, \frac{\pi}{4}]$.

Now, let $N \geq 10$. Using item 2 of Lemma 4.1 we have

$$\mathcal{Q}_{N,2}(2\sigma^2 x^2) \geq \mathcal{Q}_{N,2}(N) \geq \mathcal{Q}_{10,2}(10) > 4.65,$$

which implies

$$\left(\frac{N}{N+2} + \frac{\frac{8N}{N+1}}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)} \right) \frac{4x}{\pi^2} \leq \left(1 + \frac{8}{4.65}\right) \frac{4x}{\pi^2} \leq \frac{7x}{6}.$$

Therefore, (4.20), and thus, (4.18) hold for every $x \in (\frac{\pi}{6}, \frac{\pi}{4}]$ and $N \geq 4$. This concludes the proof of the Proposition 4.3. $\square$

Before stating the next lemma we need to introduce some notation which will be helpful in its proof. If x is an arbitrary real number, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the greatest integer less than or equal to x and the smallest integer greater than or equal to x, respectively.

Lemma 4.4. *The function*

$$g(y) = \frac{1}{1 + \frac{y}{(N+2)}} - \frac{1}{\mathcal{Q}_{N,1}(y)}$$

is increasing on the interval $(0, N+1]$.

Before we give the proof of Lemma 4.4, we first show the plots of the function g for $N = 1$, 10, and 100, respectively, in Figure 1.

Proof of Lemma 4.4. The first derivative of g is

$$g'(y) \quad = \quad \frac{\sum_{n=0}^{\infty} \frac{(n+1)(N+1)! y^n}{(N+2+n)!}}{\left(\sum_{n=0}^{\infty} \frac{(N+1)! y^n}{(N+1+n)!}\right)^2} - \frac{\frac{1}{N+2}}{(1 + \frac{y}{N+2})^2} = \frac{\frac{1}{N+2}}{(\sum_{n=0}^{\infty} \frac{(N+1)! y^n}{(N+1+n)!})^2 (1 + \frac{y}{N+2})^2} I(y),$$

where

$$I(y) := \left(1 + \frac{y}{N+2}\right)^2 \sum_{n=0}^{\infty} \frac{(n+1)(N+2)! y^n}{(N+2+n)!} - \left(\sum_{n=0}^{\infty} \frac{(N+1)! y^n}{(N+1+n)!} \right)^2 . \tag{4.22}$$

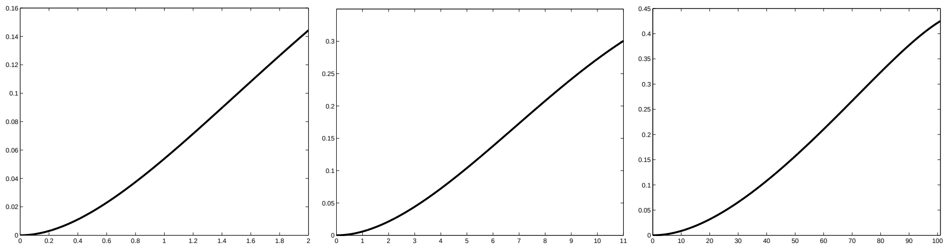

FIGURE 1. The plots of the function g in Lemma 4.4 (from left to right) with $N = 1$, 10, and 100, respectively.

Our goal will be to show $I(y) \geq 0$ for all $y \in (0, N+1]$.

First, note that the series involved in the definition of I are bounded by geometric series, because $0 < y \leq N+1$. This implies that the series in the right-hand side of eq. (4.22) converge uniformly. Now,

$$
\begin{aligned}
& \left(\sum_{n=0}^{\infty} \frac{(N+1)!y^n}{(N+1+n)!}\right)^2 \\
= & \left(1+\sum_{n=1}^{\infty} \frac{(N+1)!y^n}{(N+1+n)!}\right)\left(1+\sum_{l=1}^{\infty} \frac{(N+1)!y^l}{(N+1+l)!}\right) \\
= & \ 1+2\sum_{n=1}^{\infty} \frac{(N+1)!y^n}{(N+1+n)!}+\left(\sum_{n=1}^{\infty} \frac{(N+1)!y^n}{(N+1+n)!}\right)\left(\sum_{l=1}^{\infty} \frac{(N+1)!y^l}{(N+1+l)!}\right) \\
= & \ 1+2\sum_{n=1}^{\infty} \frac{1}{(N+2)\cdots(N+1+n)}y^n+\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(N+2)\cdots(N+1+n)}\cdot\frac{1}{(N+2)\cdots(N+1+l)}y^{n+l} \\
= & \ 1+\frac{2}{N+2}y+\frac{2}{(N+2)(N+3)}y^2+\frac{2}{(N+2)(N+3)(N+4)}y^3+2\sum_{n=4}^{\infty}\frac{1}{(N+2)\cdots(N+1+n)}y^n+ \\
& \frac{1}{(N+2)^2}y^2+\frac{2}{(N+2)^2(N+3)}y^3+\sum_{n=4}^{\infty}\left[\sum_{l=1}^{n-1}\frac{1}{(N+2)\cdots(N+1+n-l)}\cdot\frac{1}{(N+2)\cdots(N+1+l)}\right]y^n \\
= & \ 1+\frac{2}{N+2}y+\left(\frac{2}{(N+2)(N+3)}+\frac{1}{(N+2)^2}\right)y^2+\left(\frac{2}{(N+2)(N+3)(N+4)}+\frac{2}{(N+2)^2(N+3)}\right)y^3+ \\
& \sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k-1}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k-i+1)}+\left(\frac{1}{(N+2)\cdots(N+k+1)}\right)^2\right)y^{2k} \\
& +\sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k+1-i+1)}\right)y^{2k+1}\ .
\end{aligned}
$$

This expansion of the term $\left(\sum_{n=0}^{\infty}\frac{(N+1)!y^n}{(N+1+n)!}\right)^2$ now yields

$$
\begin{aligned}
& I(y) \\
= \quad & \frac{2}{N+3}y+\left(\frac{3}{(N+3)(N+4)}+\frac{2}{(N+2)(N+3)}\right)y^2 \\
& +\left(\frac{4}{(N+3)(N+4)(N+5)}+\frac{4}{(N+2)(N+3)(N+4)}\right)y^3 \\
& +\sum_{n=4}^{\infty}\left(\frac{n+1}{(N+3)\cdots(N+n+2)}+\frac{2n-2}{(N+2)\cdots(N+n+1)}\right. \\
& \left.+\frac{n-1}{(N+2)^2(N+3)\cdots(N+n)}\right)y^n \\
& -\sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k-1}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k-i+1)}\right. \\
& \left.+\left(\frac{1}{(N+2)\cdots(N+k+1)}\right)^2\right)y^{2k} \\
& -\sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k+1-i+1)}\right)y^{2k+1}\,.
\end{aligned}
$$

Using $0<y\le N+1$, one can easily verify the following inequalities

$$
\begin{aligned}
\frac{y^i}{(N+2)\cdots(N+i+1)}\cdot\frac{y^{2k-i}}{(N+2)\cdots(N+2k-i+1)} &\le \frac{y^{i-1}}{(N+3)\cdots(N+i+1)}; \\
\left(\frac{y^k}{(N+2)\cdots(N+k+1)}\right)^2 &\le \frac{y^{k-1}}{(N+3)\cdots(N+k+1)}; \\
\frac{y^i}{(N+2)\cdots(N+i+1)}\cdot\frac{y^{2k+1-i}}{(N+2)\cdots(N+2k+1-i+1)} &\le \frac{y^{i-1}}{(N+3)\cdots(N+i+1)}.
\end{aligned}
$$

The previous inequalities imply

$$
\begin{aligned}
& \left(2\sum_{i=k+1}^{2k-1}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k-i+1)}\right. \\
& \quad \left.+\left(\frac{1}{(N+2)\cdots(N+k+1)}\right)^2\right)y^{2k} \\
= \quad & 2\sum_{i=k+1}^{2k-1}\frac{y^i}{(N+2)\cdots(N+i+1)}\cdot\frac{y^{2k-i}}{(N+2)\cdots(N+2k-i+1)} \\
& +\left(\frac{y^k}{(N+2)\cdots(N+k+1)}\right)^2 \\
\le \quad & 2\sum_{i=k+1}^{2k-1}\frac{y^{i-1}}{(N+3)\cdots(N+i+1)}+\frac{y^{k-1}}{(N+3)\cdots(N+k+1)}\,.
\end{aligned}
$$

Using again $0 < y \le N+1$ we have,

$$\begin{aligned}
&\left(2\sum_{i=k+1}^{2k}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k+1-i+1)}\right)y^{2k+1}\\
\le\quad & 2\sum_{i=k+1}^{2k}\frac{y^{i-1}}{(N+2)\cdots(N+i+1)} < 2\sum_{i=k+1}^{2k}\frac{y^{i-1}}{(N+3)\cdots(N+i+1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k-1}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k-i+1)}\right.\\
&\quad\left.+\left(\frac{1}{(N+2)\cdots(N+k+1)}\right)^2\right)y^{2k}\\
\le\quad & \sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k-1}\frac{y^{i-1}}{(N+3)\cdots(N+i+1)}+\frac{y^{k-1}}{(N+3)\cdots(N+k+1)}\right)\\
=\quad & \sum_{k=2}^{\infty}\frac{y^{k-1}}{(N+3)\cdots(N+k+1)}+\sum_{i=3}^{\infty}\sum_{k=\lceil\frac{i+1}{2}\rceil}^{i-1}\frac{2y^{i-1}}{(N+3)\cdots(N+i+1)}\\
=\quad & \sum_{k=2}^{\infty}\frac{y^{k-1}}{(N+3)\cdots(N+k+1)}+\sum_{i=3}^{\infty}\frac{2(\lfloor\frac{i+1}{2}\rfloor-1)y^{i-1}}{(N+3)\cdots(N+i+1)}\\
=\quad & \frac{y}{N+3}+\sum_{k=3}^{\infty}\frac{(2\lfloor\frac{k+1}{2}\rfloor-1)y^{k-1}}{(N+3)\cdots(N+k+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(2\sum_{i=k+1}^{2k}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k+1-i+1)}\right)y^{2k+1}\\
\le\quad & \sum_{k=3}^{\infty}\frac{2\lfloor\frac{k}{2}\rfloor y^{k-1}}{(N+3)\cdots(N+k+1)}.
\end{aligned}$$

By adding the last two inequalities we obtain

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\Bigg(2\sum_{i=k+1}^{2k-1}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k-i+1)} \\
&\quad +\Big(\frac{1}{(N+2)\cdots(N+k+1)}\Big)^2\Bigg)y^{2k} \\
&\quad +\sum_{k=2}^{\infty}\Bigg(2\sum_{i=k+1}^{2k}\frac{1}{(N+2)\cdots(N+i+1)}\cdot\frac{1}{(N+2)\cdots(N+2k+1-i+1)}\Bigg)y^{2k+1} \\
\leq\ & \frac{y}{N+3}+\sum_{k=3}^{\infty}\frac{(2(\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k+1}{2}\rfloor)-1)y^{k-1}}{(N+3)\cdots(N+k+1)} \\
=\ & \frac{y}{N+3}+\sum_{k=3}^{\infty}\frac{(2k-1))y^{k-1}}{(N+3)\cdots(N+k+1)} \\
=\ & \frac{y}{N+3}+\sum_{k=2}^{\infty}\frac{(2k+1)y^{k}}{(N+3)\cdots(N+k+2)} \\
=\ & \frac{y}{N+3}+\frac{5y^2}{(N+3)(N+4)}+\frac{7y^3}{(N+3)(N+4)(N+5)}+\sum_{n=4}^{\infty}\frac{(2n+1)y^n}{(N+3)\cdots(N+n+2)}.
\end{aligned}
$$

Furthermore, it can easily be verified that

$$
\begin{aligned}
&\sum_{n=4}^{\infty}\Bigg(\frac{n+1}{(N+3)\cdots(N+n+2)}+\frac{2n-2}{(N+2)\cdots(N+n+1)} \\
&\quad +\frac{n-1}{(N+2)^2(N+3)\cdots(N+n)}\Bigg)y^n \\
\geq\ & \sum_{n=4}^{\infty}\frac{(4n-2)y^n}{(N+3)\cdots(N+n+2)} \\
\geq\ & \sum_{n=4}^{\infty}\frac{(2n+1)y^n}{(N+3)\cdots(N+n+2)}.
\end{aligned}
$$

Combining the previous inequalities we readily obtain

$$I(y)\geq 0,\quad y\in(0,N+1],$$

so, $g'(y)\geq 0$, for all $y\in(0,N+1]$. □

LEMMA 4.5. *For every $x\in[\frac{\pi}{4},\frac{\pi}{3}]$, define*

$$h_x(a)=\frac{1}{x(1+ax^2)}-\frac{2}{(\pi-2x)(1+a(\pi-2x)^2)}.$$

Then, h_x is an increasing function on the interval $[\frac{4}{3\pi^2},\frac{4}{\pi^2}]$.

PROOF. Routine calculations yield

$$\begin{aligned} h'_x(a) &= -\frac{x}{(1+ax^2)^2} + \frac{2(\pi-2x)}{(1+a(\pi-2x)^2)^2} \\ &= \frac{2xa(\pi-2x)(2x-(\pi-2x)) + 2(\pi-2x) - x + a^2x(\pi-2x)(2x^3-(\pi-2x)^3)}{(1+ax^2)^2(1+a(\pi-2x)^2)^2} \\ &\geq \frac{2(\pi-2x) - x + a^2x(\pi-2x)(2x^3-(\pi-2x)^3)}{(1+ax^2)^2(1+a(\pi-2x)^2)^2}. \end{aligned}$$

First, observe that if $x \in [0, \frac{\pi}{3}]$, then $2(\pi-2x) - x \geq \pi/3$, and if $x \in [\frac{\pi}{2+2^{1/3}}, \frac{\pi}{3}]$, then $2x^3 - (\pi-2x)^3 \geq 0$, and thus

$$2(\pi-2x) - x + a^2x(\pi-2x)(2x^3-(\pi-2x)^3) > 0.$$

So, regardless of the values of a, $h'_x(a)$ is positive for every $x \in [\frac{\pi}{2+2^{1/3}}, \frac{\pi}{3}]$.

Now, assume $x \in [\frac{\pi}{4}, \frac{\pi}{2+2^{1/3}}]$. Then, $2x^3 - (\pi-2x)^3 < 0$, so, if $a \in [\frac{4}{3\pi^2}, \frac{4}{\pi^2}]$, then

$$2(\pi-2x) - x + a^2x(\pi-2x)(2x^3-(\pi-2x)^3) \geq \omega(x),$$

where $\omega(x) := 2(\pi-2x) - x + \frac{16}{\pi^4}x(\pi-2x)(2x^3-(\pi-2x)^3)$. The proof of the lemma will be completed, once we establish $\omega(x) > 0$, for every x in $[\frac{\pi}{4}, \frac{\pi}{2+2^{1/3}})$.

First, note that $\omega(\pi/4) > 0$. On the other hand, $\omega'(x) \geq \frac{8}{\pi^2}[x^2 + (2x-5)^2]$. But the quadratic polynomial $x^2 + (2x-5)^2$ is decreasing on $(-\infty, 2)$, thus ω' restricted on $[\frac{\pi}{4}, \frac{\pi}{2+2^{1/3}}]$ attains its minimum value at $x = \frac{\pi}{2+2^{1/3}}$. Since, $\frac{\pi}{2+2^{1/3}} < 1$, we obtain

$$\omega'\left(\frac{\pi}{2+2^{1/3}}\right) > \omega'(1) \geq 5\left(\frac{16}{\pi^2} - 1\right) > 0.$$

This establishes that ω is increasing in the interval $[\frac{\pi}{4}, \frac{\pi}{2+2^{1/3}}]$, and since $\omega(\pi/4) > 0$, we finally conclude that $h'_x(a) > 0$, for all $x \in [\frac{\pi}{4}, \frac{\pi}{2+2^{1/3}}]$ and $a \in [\frac{1}{3}(\frac{4}{\pi^2}), \frac{4}{\pi^2}]$. Thus, h_x is an increasing function on the interval $[\frac{1}{3}(\frac{4}{\pi^2}), \frac{4}{\pi^2}]$. □

LEMMA 4.6. *Let*

$$p(y) = \frac{1}{y^3} - \frac{1}{\sin^2 y \tan y} = \frac{\frac{\sin^2 y \tan y}{y^3} - 1}{\sin^2 y \tan y}$$

and

$$q(y) = \frac{\sin^2 y \tan y}{y^3} - 1.$$

Then, for every $y \in [\frac{\pi}{8}, \frac{\pi}{4}]$ we have $p(y) \geq 0$ and that q is increasing.

PROOF. Obviously,

$$p(y) = \frac{q(y)}{\sin^2 y \tan y}.$$

It is also easy to verify

$$q'(y) = \frac{2y\sin^2 y + y\tan^2 y - 3\sin^2 y \tan y}{y^4} = \frac{\tan^2 y(2y\cos^2 y + y - \frac{3}{2}\sin 2y)}{y^4}.$$

Now, let

$$r(y) = 2y\cos^2 y + y - \frac{3}{2}\sin 2y.$$

Then,

$$\begin{aligned} r'(y) &= 2\cos^2 y - 4y\cos y\sin y + 1 - 3\cos 2y = 2 - 2\cos 2y - 2y\sin 2y, \\ r''(y) &= 4\sin 2y - 2\sin 2y - 4y\cos 2y = 2\cos 2y(\tan 2y - 2y). \end{aligned}$$

Clearly, $r''(y) \geq 0$ for all $y \in [0, \frac{\pi}{4}]$ and $r(0) = r'(0) = 0$. Thus, $r'(y) > 0$ and so $r(y) > 0$, for every $y \in [0, \frac{\pi}{4}]$. Therefore,

$$q'(y) \geq 0, \quad y \in \left(0, \frac{\pi}{4}\right].$$

Since $\lim_{y\to 0+} q(y) = 0$ we have $q(y) > 0$ for every $y \in [0, \frac{\pi}{4}]$, so $p(y) > 0$ for all $y \in [\frac{\pi}{8}, \frac{\pi}{4}]$. □

Lemma 4.7. *The function*

$$D(x) = \left(\frac{1}{x} - \frac{1}{2\tan\frac{x}{2}}\right) - \frac{\frac{4x}{3\pi^2}}{1 + \frac{4x^2}{3\pi^2}} + 2\cdot\frac{\frac{4(\pi-2x)}{3\pi^2}}{1 + \frac{4(\pi-2x)^2}{3\pi^2}} - 2\left(\frac{1}{(\pi - 2x)} - \frac{1}{2\tan\frac{(\pi-2x)}{2}}\right)$$

is non-negative on the interval $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$.

Proof. Elementary computations give

$$D\left(\frac{\pi}{4}\right) > 0.013, \quad D\left(\frac{\pi}{3}\right) > 0.034.$$

In view of these numerical results, the conclusion the lemma will be established, once we show $D''(x) \leq 0$ for $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$, because, then D is concave downward.

Routine calculations give

$$\begin{aligned} D'(x) &= -\frac{\frac{4}{3\pi^2}(1 - \frac{4x^2}{3\pi^2})}{(1 + \frac{4x^2}{3\pi^2})^2} - 4\cdot\frac{\frac{4}{3\pi^2}(1 - \frac{4(\pi-2x)^2}{3\pi^2})}{(1 + \frac{4(\pi-2x)^2}{3\pi^2})^2} \\ &\quad + \left(-\frac{1}{x^2} + \frac{1}{4\sin^2\frac{x}{2}}\right) + 4\left(-\frac{1}{(\pi - 2x)^2} + \frac{1}{4\sin^2\frac{(\pi-2x)}{2}}\right) \end{aligned}$$

and

$$\begin{aligned} D''(x) &= \frac{2(\frac{4}{3\pi^2})^2 x(3 - \frac{4x^2}{3\pi^2})}{(1 + \frac{4x^2}{3\pi^2})^3} - 8\left(\frac{2(\frac{4}{3\pi^2})^2(\pi - 2x)(3 - \frac{4(\pi-2x)^2}{3\pi^2})}{(1 + \frac{4(\pi-2x)^2}{3\pi^2})^3}\right) \\ &\quad + \left(\frac{2}{x^3} - \frac{1}{4\sin^2\frac{x}{2}\tan\frac{x}{2}}\right) - 8\left(\frac{2}{(\pi - 2x)^3} - \frac{1}{4\sin^2\frac{(\pi-2x)}{2}\tan\frac{(\pi-2x)}{2}}\right) \\ &= \frac{\frac{32}{9\pi^4}x(3 - \frac{4x^2}{3\pi^2})}{(1 + \frac{4x^2}{3\pi^2})^3} - 8\left(\frac{\frac{32}{9\pi^4}(\pi - 2x)(3 - \frac{4(\pi-2x)^2}{3\pi^2})}{(1 + \frac{4(\pi-2x)^2}{3\pi^2})^3}\right) \\ &\quad + \frac{1}{4}\left(\frac{1}{(\frac{x}{2})^3} - \frac{1}{\sin^2\frac{x}{2}\tan\frac{x}{2}}\right) - 2\left(\frac{1}{(\frac{\pi-2x}{2})^3} - \frac{1}{\sin^2\frac{(\pi-2x)}{2}\tan\frac{(\pi-2x)}{2}}\right). \end{aligned}$$

For $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$, we have $\frac{\pi}{3} \leq \pi - 2x \leq \frac{\pi}{2}$. A routine first-order derivative test shows that $a(y) := y(3 - \frac{4y^2}{3\pi^2})$ is increasing on $[\frac{\pi}{4}, \frac{\pi}{2}]$. Applying this result both for the restriction of a on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$ and the composite function $x \to a(\pi - 2x)$, where $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$, we obtain,

$$\begin{aligned} & \frac{\frac{32}{9\pi^4} x(3 - \frac{4x^2}{3\pi^2})}{(1 + \frac{4x^2}{3\pi^2})^3} - 8 \left(\frac{\frac{32}{9\pi^4}(\pi - 2x)(3 - \frac{4(\pi-2x)^2}{3\pi^2})}{(1 + \frac{4(\pi-2x)^2}{3\pi^2})^3} \right) \\ \leq \; & \frac{\frac{32}{9\pi^4} \frac{\pi}{3}(3 - \frac{4}{3\pi^2} \cdot \frac{\pi^2}{9})}{(1 + \frac{4}{3\pi^2} \cdot \frac{\pi^2}{16})^3} - 8 \left(\frac{\frac{32}{9\pi^4} \frac{\pi}{3}(3 - \frac{4}{3\pi^2} \cdot \frac{\pi^2}{9})}{(1 + \frac{4}{3\pi^2} \cdot \frac{\pi^2}{4})^3} \right) \\ < \; & -0.2\ . \end{aligned}$$

Moreover, Lemma 4.6, implies

$$-2 \left(\frac{1}{(\frac{\pi-2x}{2})^3} - \frac{1}{\sin^2 \frac{(\pi-2x)}{2} \tan \frac{(\pi-2x)}{2}} \right) \leq 0$$

and

$$\frac{1}{4} \left(\frac{1}{(\frac{x}{2})^3} - \frac{1}{\sin^2 \frac{x}{2} \tan \frac{x}{2}} \right) = \frac{\frac{\sin^2 \frac{x}{2} \tan \frac{x}{2}}{(\frac{x}{2})^3} - 1}{4 \sin^2 \frac{x}{2} \tan \frac{x}{2}} \leq \frac{\frac{\sin^2 \frac{\pi}{6} \tan \frac{\pi}{6}}{(\frac{\pi}{6})^3} - 1}{4 \sin^2 \frac{\pi}{8} \tan \frac{\pi}{8}} < 0.03.$$

Combining all the previous inequalities we conclude

$$D''(x) \leq -0.2 + 0.03 + 0 < 0.$$

□

Eq. (3.1) in the previous section implies that L_N is continuous on $[0, \pi]$. Therefore, in order to prove Theorem 1.4 it is enough to restrict our discussion on the interval $(0, \pi)$.

Proof of item 2 of Theorem 1.4. Since L_N is even and

$$|L_N(\xi) L_N(2\xi)| = |L_N(\xi) L_N(2\pi - 2\xi)| = L_N(\xi) L_N(2\pi - 2\xi)\ .$$

Changing the variable from $\pi - \xi$ to x, we assert that $L_N(\xi) L_N(2\xi)$ is decreasing on $[\frac{2\pi}{3}, \pi]$ if and only if $L_N(\pi - x) L_N(2x)$ is increasing on $[0, \frac{\pi}{3}]$, equivalently

$$(\ln (L_N(\pi - x) L_N(2x)))' \geq 0,$$

for every $0 \leq x \leq \frac{\pi}{3}$.

Since, $\frac{2\pi}{3} \leq \pi - x \leq \pi$ for every $x \in [0, \frac{\pi}{3}]$, we get

$$L_N(\pi - x) = \frac{\sqrt{1 - \widehat{h}^2_{N,\sigma}(x)}}{\sin^{N+1}(\frac{x}{2})}.$$

We will give the proof of above inequality on the intervals $(0, \frac{\pi}{4})$ and $[\frac{\pi}{4}, \frac{\pi}{3}]$, separately.

First, let $x \in (0, \frac{\pi}{4})$. Then, $0 < 2x < \frac{\pi}{2}$. Therefore,

$$L_N(2x) = \frac{\widehat{h}_{N,\sigma}(2x)}{\cos^{N+1}(x)}.$$

So, if $x \in (0, \frac{\pi}{4})$, routine calculations give

$$\frac{[\ln (L_N(\pi - x) L_N(2x))]'}{N+1} = \frac{1}{\frac{x}{2}(2 + \mathcal{Q}_{N,3}(\frac{\sigma^2 x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2 x^2}{2})} - \frac{1}{2 \tan \frac{x}{2}} + \tan x - \frac{(\frac{4\sigma^2}{N+1})x}{\mathcal{Q}_{N,2}(2\sigma^2 x^2)}.$$

A direct application of Proposition 4.3, implies

$$\frac{(\ln L_N(\pi - x)L_N(2x))'}{N+1} \geq 0.$$

Therefore, $L_N(\pi - x)L_N(2x)$ is increasing on $[0, \frac{\pi}{4}]$.

We now focus on $x \in [\frac{\pi}{4}, \frac{\pi}{3})$. Obviously, all such x satisfy $\frac{\pi}{2} \leq 2x < \frac{2\pi}{3}$. Therefore,

$$L_N(2x)L_N(\pi - x) = \frac{\sqrt{1 - \widehat{h}^2_{N,\sigma}(\pi - 2x)}}{\sin^{N+1}(\frac{\pi - 2x}{2})} \frac{\sqrt{1 - \widehat{h}^2_{N,\sigma}(x)}}{\sin^{N+1}(\frac{x}{2})}.$$

Thus,

$$\begin{aligned} \frac{(\ln L_N(\pi - x)L_N(2x))'}{N+1} &= \frac{1}{\frac{x}{2}(2 + \mathcal{Q}_{N,3}(\frac{\sigma^2 x^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2 x^2}{2})} - \frac{1}{2\tan\frac{x}{2}} \\ &\quad - \frac{2}{\frac{\pi - 2x}{2}(2 + \mathcal{Q}_{N,3}(\frac{\sigma^2(\pi-2x)^2}{2}))\mathcal{Q}_{N,1}(\frac{\sigma^2(\pi-2x)^2}{2})} + \frac{1}{\tan\frac{\pi - 2x}{2}}. \end{aligned}$$

We denote the right-hand-side of above equation as I. Using the definition of $\mathcal{Q}_{N,4}$, we rewrite I:

$$I = I_1 + I_2 + I_3 \ ,$$

where

$$\begin{aligned} I_1 &= \frac{2}{\pi - 2x}\mathcal{Q}_{N,4}\left(\frac{\sigma^2(\pi - 2x)^2}{2}\right) - \frac{1}{x}\mathcal{Q}_{N,4}\left(\frac{\sigma^2 x^2}{2}\right), \\ I_2 &= \frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2 x^2}{2})} - \frac{2}{(\pi - 2x)\mathcal{Q}_{N,1}(\frac{\sigma^2(\pi-2x)^2}{2})}, \\ I_3 &= -\frac{1}{2\tan\frac{x}{2}} + \frac{1}{\tan\frac{\pi-2x}{2}}. \end{aligned}$$

Now, observe that for every $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$ we have,

$$x \leq \pi - 2x \leq \pi/2 \ ,$$

and thus,

$$\frac{1}{x} \leq \frac{2}{\pi - 2x}. \tag{4.23}$$

Moreover, due to Theorem 1.1, we obtain $\frac{\sigma^2 x^2}{2} < \gamma_N < N+1$ and $\frac{\sigma^2(\pi-2x)^2}{2} < \gamma_N < N+1$, for all $N \geq 1$ and $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$; so, by (4.23) and item 3 of Lemma 4.1 we conclude $I_1 \geq 0$.

An elementary calculation establishes

$$\begin{aligned} I_2 &= \frac{1}{x\mathcal{Q}_{N,1}(\frac{\sigma^2 x^2}{2})} - \frac{1}{x(1 + \frac{\sigma^2 x^2}{2(N+2)})} + \frac{1}{x(1 + \frac{\sigma^2 x^2}{2(N+2)})} - \frac{2}{(\pi - 2x)\mathcal{Q}_{N,1}(\frac{\sigma^2(\pi-2x)^2}{2})} \\ &\quad + \frac{2}{(\pi - 2x)(1 + \frac{\sigma^2(\pi-2x)^2}{2(N+2)})} - \frac{2}{(\pi - 2x)(1 + \frac{\sigma^2(\pi-2x)^2}{2(N+2)})}. \end{aligned}$$

Using g defined in Lemma 4.4, we obtain

$$I_2 = \frac{2}{\pi - 2x}g\left(\frac{\sigma^2(\pi - 2x)^2}{2}\right) - \frac{1}{x}g\left(\frac{\sigma^2 x^2}{2}\right) + \frac{1}{x(1 + \frac{\sigma^2 x^2}{2(N+2)})} - \frac{2}{(\pi - 2x)(1 + \frac{\sigma^2(\pi-2x)^2}{2(N+2)})}.$$

Combining $\frac{\sigma^2 x^2}{2} < N+1$ and $\frac{\sigma^2(\pi-2x)^2}{2} < N+1$, (4.23) and the fact that g is increasing on $[0, N+1)$, for all $N \geq 1$,

$$I_2 \geq \frac{1}{x(1+\frac{\sigma^2 x^2}{2(N+2)})} - \frac{2}{(\pi-2x)(1+\frac{\sigma^2(\pi-2x)^2}{2(N+2)})}\,. \tag{4.24}$$

Using once again $\gamma_N < N+1$, for every $N \geq 1$, and item 2 of Theorem 1.1 we have

$$\frac{4}{3\pi^2} \leq \frac{\sigma^2}{2(N+2)} \leq \frac{4}{\pi^2}, \quad N \geq 1.$$

Setting $a = \frac{\sigma^2}{2(N+2)}$ and then applying Lemma 4.5, inequality (4.24) implies,

$$I_2 \geq \frac{1}{x(1+\frac{4x^2}{3\pi^2})} - \frac{2}{(\pi-2x)(1+\frac{4(\pi-2x)^2}{3\pi^2})}.$$

Combining the previous inequality with $I_1 \geq 0$, we have

$$\begin{aligned} I &\geq \frac{1}{x(1+\frac{4x^2}{3\pi^2})} - \frac{1}{2\tan\frac{x}{2}} + \frac{1}{\tan\frac{(\pi-2x)}{2}} - \frac{2}{(\pi-2x)(1+\frac{4(\pi-2x)^2}{3\pi^2})} \\ &= \left(\frac{1}{x} - \frac{1}{2\tan\frac{x}{2}}\right) - \frac{\frac{4x}{3\pi^2}}{1+\frac{4x^2}{3\pi^2}} + 2\cdot\frac{\frac{4(\pi-2x)}{3\pi^2}}{1+\frac{4(\pi-2x)^2}{3\pi^2}} - 2\left(\frac{1}{(\pi-2x)} - \frac{1}{2\tan\frac{(\pi-2x)}{2}}\right), \end{aligned}$$

for $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$. Finally, Lemma 4.7 readily implies,

$$I \geq 0,$$

which completes the proof of second part of Theorem 1.4. □

We will conclude the present section with the proof of Theorem 1.5.

Proof of Theorem 1.5. According to the remarks preceding the statement of Theorem 1.5 the only conclusion that remains to be proved is $\lim_{N\to\infty}(N - b_N) = \infty$.

The definition of b_N implies

$$b_N = \log_2 \sqrt{1 - \widehat{h}^2_{N,\sigma}\left(\frac{\pi}{3}\right)} + N + 1\,, \tag{4.25}$$

for every $N \geq 1$.

In order to establish $\lim_{N\to+\infty}(N - b_N) = +\infty$ it suffices to prove

$$\lim_{N\to+\infty} \widehat{h}_{N,\sigma}(\pi/3) = 1 \tag{4.26}$$

Using g_N defined in the proof of Theorem 1.1 we infer

$$\widehat{h}_{N,\sigma}\left(\frac{\pi}{3}\right) = g_N\left(\frac{4}{9}\gamma_N\right)\,, \tag{4.27}$$

where γ_N is defined so that $\widehat{h}_{N,\sigma}(\frac{\pi}{2}) = g_N(\gamma_N) = \frac{\sqrt{2}}{2}$. Recall that g_N is decreasing (refer again to the proof of Theorem 1.1). By Theorem 1.1 $\gamma_N < N$ for all $N \geq 2$. Thus,

$$g_N\left(\frac{4}{9}N\right) < g_N\left(\frac{4}{9}\gamma_N\right) < 1\,.$$

In order to prove (4.26), it is enough to show $\lim_{N\to+\infty} g_N(\frac{4}{9}N) = 1$. In fact, a more general result is true: $\lim_{N\to+\infty} g_N(xN) = 1$, for $0 < x < 1$. Let us now establish this result.

Fix $0 < x < 1$. Using the definition of g_N we have

$$\begin{aligned} 1 - g_N(xN) &= e^{-Nx} \sum_{n=N+1}^{\infty} \frac{(Nx)^n}{n!} \\ &= e^{-Nx} \frac{(Nx)^{N+1}}{(N+1)!} \left(1 + \sum_{k=1}^{\infty} \frac{(Nx)^k}{(N+2)\cdots(N+1+k)} \right) . \end{aligned}$$

On the other hand the series $\sum_{k=1}^{\infty} \frac{(Nx)^k}{(N+2)\cdots(N+1+k)}$ is dominated by the geometric series $\sum_{k=1}^{\infty} x^k$, which converges since $0 < x < 1$. Therefore,

$$0 \le 1 - g_N(xN) < e^{-Nx} \frac{(Nx)^{N+1}}{(N+1)!} (1-x)^{-1} .$$

Using Stirling's formula ($n! > e^{-n} n^n$, for all $n \in \mathbb{N}$) for $n = N+1$, we now obtain

$$e^{-Nx} \frac{(Nx)^{N+1}}{(N+1)!} (1-x)^{-1} < e^{-Nx} \frac{(Nx)^{N+1} e^{N+1}}{(N+1)^{N+1}} (1-x)^{-1} ,$$

which finally gives

$$0 \le 1 - g_N(xN) < (xe^{1-x})^N ex(1-x)^{-1} .$$

Observing that the function $x \to xe^{1-x}$ is increasing on $[0,1]$ we have $xe^{1-x} < 1$, for every $0 < x < 1$, which establishes $\lim_{N\to+\infty}(1 - g_N(xN)) = 0$. $\square$

5. Acknowledgments

This research was supported in part by the following grants: Texas Higher Eduaction Coordinating board ARP 003652-0071-2001, NSF DMS-0070376, NSF CHE-0074311 and R.A. Welch Foundation E-0608. Partial finacial support was also provided by the Texas Learning and Computation Center of the University of Houston and by a University of Houston GEAR 2002 grant. We finally wish to express our most sincere thanks to Professor Q. Sun, currently with the University of Houston's Mathematics Department, for his valuable suggestions that helped us to improve the original manuscript.

References

[AS72] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions*, Dover Publications Inc., NY 1972.

[CG99] C. Chandler and A. G. Gibson, *Uniform approximation of functions with discrete approximation functionals*, J. Approximation Theory **100** (1999), 233–250.

[CR95] A. Cohen and R. D. Ryan, *Wavelets and Multirate Signal Processing*, Chapman & Hall, 1995.

[FS01] A. Fan, Q. Sun, *Regularity of Butterworth refinable functions*, Asian J. Math. **5**(3), pp. 433-440, 2001.

[HK93] D. K. Hoffman and D. J. Kouri, *Distributed approximating function theory for an arbitrary number of particles in a coordinate system-independent formalism*, J. Phys. Chem. **97** (1993), 4984–4988.

[HNSK91] D. K. Hoffman, N. Nayar, O. A. Sharafeddin, and D. J. Kouri, *On an analytic banded approximation for the discretized free propagator*, J. Phys. Chem. **95** (1991), 8299–8305.

[HW96] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, 1996.

[HE95] , L. Herve, *Construction et regularite des fonctions d' echelle*, SIAM J. Math. Anal., **26**, pp. 1361-1385, 1995.

[KPS1] I.A. Kakadiaris, M. Papadakis, L. Shen, D.J. Kouri, and D.J. Hoffman, *g-HDAF multiresolution deformable models for shape modeling and reconstruction*, British Machine Vision Conference (Cardiff, UK), Sep 2002.

[KPS2] ———, *m-HDAF multiresolution deformable models*, The 14th International Conference on Digital Signal Processing (Santorini, Greece), Jul 2002.

[Mal89] S.G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $l^2(\vec{r})$, Trans. Amer. Math. Soc. **315** (1989), 69–87.

[S99] Q. Sun, *Sobolev exponent estimates and asymptotic regularity of the M-band Daubechies' scaling functions*, Constr. Approx., **15**, pp. 441-465, 1999.

[V94] L.F. Villemoes, *Wavelet analysis of refinement equations*, SIAM J. Math. Anal., **25**, pp. 1433-1466, 1994.

[WWH01] G. Wei, H. Wang, D. K. Hoffman, D.J. Kouri, I. A. Kakadiaris, and M. Papadakis, *On the mathematical properties of distributed approximating functionals*, J. Phys. Chem. **30** (2001), no. 1, 83–107.

School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China
E-mail address: `yjs@math.pku.edu.cn`

Department of Mathematics, West Virginia University, Morgantown, WV 26506
E-mail address: `lshen@math.wvu.edu`

Department of Mathematics, University of Houston, Houston, TX 77204-3008
E-mail address: `mpapadak@math.uh.edu`

Department of Computer Science, University of Houston, Houston, TX 77204-3008
E-mail address: `ioannisk@uh.edu`

Department of Physics, University of Houston, Houston, TX 77204
E-mail address: `Kouri@uh.edu`

Department of Chemistry, Iowa State University, Ames, Iowa 50011
E-mail address: `hoffman@ameslab.edu`