# Examples of Inverse problems

- **Linear system** Given $A \in M_{n\times n}(\mathbb{R})$ and $x \in \mathbb{R}^n$.

- Solving to find $Ax$ $\longrightarrow$ Direct problem

- Solving for $x$ in $Ax = y$ for some given $y$ is the inverse problem.

- **Direct problem**

Find zeros of a given polynomial of degree $n$.

**Inverse problem**

Find the polynomial $p$ of degree $n$ with given zeros $x_1, \ldots, x_n$.

- **Direct problem**

Calculate the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x_1, \ldots, x_n \in \mathbb{R}$.

**Inverse problem**

Find a polynomial $p$ of degree $n$ that assummes given values $y_1, \ldots, y_n \in \mathbb{R}$ at given points $x_1, \ldots, x_n \in \mathbb{R}$.

- Inteligence test

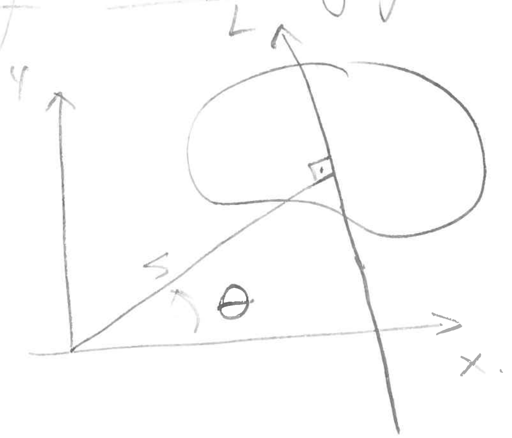
Direct problem

Evaluate the sequence $\{a_n\}_{n \in \mathbb{N}}$ given the law of formation.

Inverse problem

Given the first few terms $a_1, a_2, \ldots, a_k$ in a sequence find the law of formation of the sequence.

- Computer tomography. { Nobel to 1979 Allan Cormack and Godfrey Hounsfield




$\rho(x,y)$ — density at point $(x,y)$.

Characterization of line $L$

$y = mx + b$ — characterized by all pairs $(m,b)$.

But this does not include vertical lines!

Beter try point-normal characterization!

If $\vec{n} \perp L$ then there exists $\theta \in [0, 2\pi)$

Such that $\vec{n} \parallel$ line radiating out from origin. at an angle $\theta$ measured counterclock from the positive x-axis

Intersection of this line with $L$ at $(s\cos\theta, s\sin\theta)$

Note that $L = L_{s,\theta}$. Since $L_{s,\theta+2\pi} = L_{s,\theta}$

and $L_{s,\theta+\pi} = L_{-s,\theta}$ for all $\theta$ chose.

$$\boxed{l_{s,\theta} \text{ with } s \in \mathbb{R} \text{ and } 0 \leq \theta < \pi.}$$

Observe that any point on is given by

$$(x, y) = (s\cos\theta, s\sin\theta) + u \cdot \langle -\sin\theta, \cos\theta \rangle, \ u \in \mathbb{R}$$

$$x(u) = s\cos\theta - u\sin\theta, \quad y(u) = s\sin\theta + u\cos\theta, \ u \in \mathbb{R}$$

or in complex notation.

$$s e^{i\theta} + i u e^{i\theta} \in \mathbb{C}, \ u \in \mathbb{R}$$

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X-rays and Beer's law

Assume that x-rays are monochromatic

- Each photon has the same energy level
- The beam propagates at fixed frequency.

- Same number of photons / second passing through every centimeter of path of the beam.

$N(x)$ – number of photons per second passing through point x

$$I(x) = E \cdot N(x).$$

↳ intensity of the beam at x

[ X-ray ~~zero~~ width and not subjected to refraction or diffraction.

- Attenuation Coefficient A.

Every substance through which an X-ray passes absorbs a proportion of the photons in the beam.

Bone – high attenuation coefficient

Air – low attenuation coefficient.

$$\text{Hounsfield unit} = \frac{A_{medium} - A_{water}}{A_{water}}$$

X-ray passes through a medium between $x$ and $x + \Delta x$. with attenuation coefficient $A(x)$.

Proportion of absorbed photons.

$$P(x) = A(x) \cdot \Delta x$$

Number of photons absorbed/second is

$$P(x) \cdot N(x) = A(x)\, N(x) \cdot \Delta x$$

Loss of intensity of the x-ray.

$$\Delta I \approx -A(x)\, I(x) \cdot \Delta x$$

Beer's law

$$\frac{dI}{dx} = -A(x)\, I(x).$$

From $\underbrace{I(x_0) = I_0}_{\text{initial intensity}}$ and $\underbrace{I_1 = I(x_1)}_{\text{Measured intensity}}$

we get:

$$\int_{x_0}^{x_1} A(x)\,dx = \ln \frac{I_0}{I_1}$$

2D CT problem

Find $A(x,y)$ given

$$\int_{L_{s,\theta}} A(x,y)\,d\sigma = \underbrace{\int_{-\infty}^{\infty} A(s\cos\theta - v\sin\theta,\ s\sin\theta + v\cos\theta)\,dv}_{\text{Radon Transform.}}$$

for every line $L_{s,\theta}$

- Electrical Impedance tomography

Quasistatics EM or stationary heat

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } D. \qquad (1)$$

Direct problem

Solve (1) with given $\gamma$ and boundary data on $\partial D$

Inverse problem

Measure $u$ and $\gamma \dfrac{\partial u}{\partial \nu}$ on $\partial D$ and determine $\gamma$ in $D$.

Inverse problem

Given the property of a linear device and the output signal find the input signal.

For example.

$$\int_D K(x,y)\,u(y)\,dy = f(x)$$

Given $k(x,y)$ and $f$ find $u$

In practice the output measured signal $f(x)$ is noisy, i.e., we measure $f_\delta$ with

$$\|f_\delta - f\| \le \delta$$

## Inverse problem

Determine the support of $D$ from the far field data $u_\infty(\hat{x})$ for all $\hat{x} \in \mathbb{R}^N$

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In general the far field pattern

$$u_\infty = A(\hat{x}, k, \hat{\theta}).$$

## Inverse spectral problems

String $L$, with density $\rho(x) > 0$ be fixed.

$V(x,t)$ - displacement at time $t$ and posit. $x$.

$$(2)\quad \begin{cases} \rho(x) \dfrac{\partial^2 V}{\partial t^2} = \dfrac{\partial^2 V}{\partial x^2} & 0 < x < L,\ t > 0 \\ V(0,t) = V(L,t) = 0 & \text{fn } t > 0 \end{cases}$$

$$\tilde{V}(x,t) = W(x)\left[ a\cos\omega t + b\sin\omega t \right]$$

solves (2) iff

$$\begin{cases} W''(x) + \omega^2 \rho(x) W(x) = 0, & 0 < x < L \\ W(0) = 0 = W(L) \end{cases}$$

$\hookrightarrow$ Sturm - Liouville eigenvalue problem

Direct problem

Compute the $\omega$, $\chi$ from $\rho$.

Inverse problem

Determine the mass density from some measured frequencies!

- Pompeiu problem. (1929)

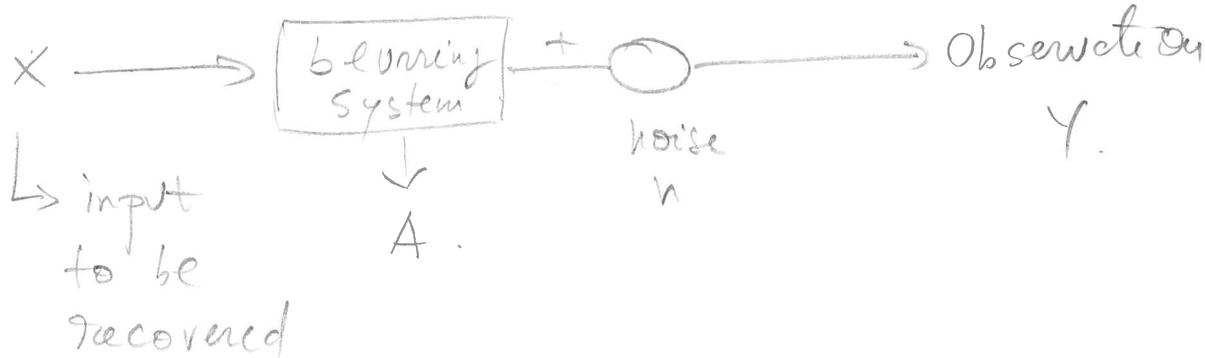
$0 \not\equiv f \in L^1_{loc}(\mathbb{R}^n)$

Assume

$$\int_D f(gx+\gamma)\,dx = 0 \quad \forall g \in SO(n), \forall \gamma \in \mathbb{R}^n, n \geq 2$$

Show that $D$ is a ball.

- <u>Image processing</u>




$$Y = Ax + n.$$

Find $x$ given $Y$, $A$.

## Inverse scattering problem

Find the shape of a scattering object given the knowledge (through measurement) of its acoustic or electromagnetic echo!

$D \subset \mathbb{R}^N$ – scatterer with smooth boundary.

Plane incident wave

$$U_i(x) = e^{ik\hat{\theta}\cdot x}$$ where $k>0$ $\left(k=\frac{w}{c}\right)$ denotes the wave number and $\hat{\theta}$ describes the direction of incident wave

Direct problem

Find $U = U_i + U_s$ where

$$\begin{cases} \Delta U + k^2 U = 0 \text{ in } \mathbb{R}^N \setminus \overline{D}, \\ U = 0 \quad \text{on } \partial D \\ \dfrac{\partial U_s}{\partial r} - ik U_s = O\left(r^{-\frac{(N+1)}{2}}\right) \text{ for } r=|x|\to\infty \text{ uniformly in } \dfrac{x}{|x|} \end{cases}$$

↳ Radiation condition

We have

$$U_s(x) = \frac{e^{ik|x|}}{|x|^{\frac{N-1}{2}}} U_\infty(\hat{x}) + O\left(|x|^{-\frac{N+1}{2}}\right) \text{ as } |x|\to\infty$$

$$\hat{x} = \frac{x}{|x|}$$

## Inverse problem

Determine the support of D from the far field data $u_\infty(\hat{x})$ for all $\hat{x} \in \mathbb{R}^N$

## Inverse source problems

$$(\nabla^2 + k^2) u = f \quad \text{in } \mathbb{R}^3$$

compact support.

$$\lim_{r \to \infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0. \quad \text{Uniformly in } \frac{x}{|x|}$$

$$u = A\,\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right).$$

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Given the radiation pattern. find f.

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## Abel Integral equation

Let a mass element move along a curve $\Gamma$ from point $p_1$ on level $h > 0$ to point $p_0$ on level $h = 0$.

Force = mg

## Direct problem

Determine the time $T$ in which the element moves from $p_1$ to $p_0$, when the curve $\Gamma$ is given.

## Inverse problem

From measured time for some values of $h$, $T(h)$ determine $\Gamma$.

$\Gamma$: $x = \psi(y)$, $P = (\psi(y), y)$.

$$E + U = \frac{m}{2}v^2 + mgy = \text{const.} = mgh. \quad (1)$$

$$(E+U)(t) = (E+U)(0)$$

$$\frac{ds}{dt} = v - \text{velocity along } \Gamma. \quad (2)$$

$$v = \sqrt{2g(h-y)} \quad \text{— from (1)}$$

from (2)

$$T = T(h) = \int_{p_0}^{p_1} \frac{ds}{v} = \int_0^h \frac{\overbrace{\sqrt{1+\psi'^2(y)}\,dy}^{ds}}{\sqrt{2g(h-y)}}$$

Abel equation $\displaystyle\int_0^h \frac{\phi(y)}{\sqrt{h-y}}\,dy = f(h)$ for $\underline{\underline{h>0}}$!

# Backwards heat equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t).$$

$$u(0,t) = u(\pi,t) = 0 \qquad t \geq 0.$$

$$u(x,0) = u_0(x) \qquad 0 \leq x \leq \pi$$

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Sep of variables

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx) \quad \text{with} \quad a_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin(ny)\,dy$$

Direct problem: Solve for $u$, given $u_0$ and $T$: final time.

Inverse problem

From measured $u(\cdot,T)$ determine $u(\cdot,0)$

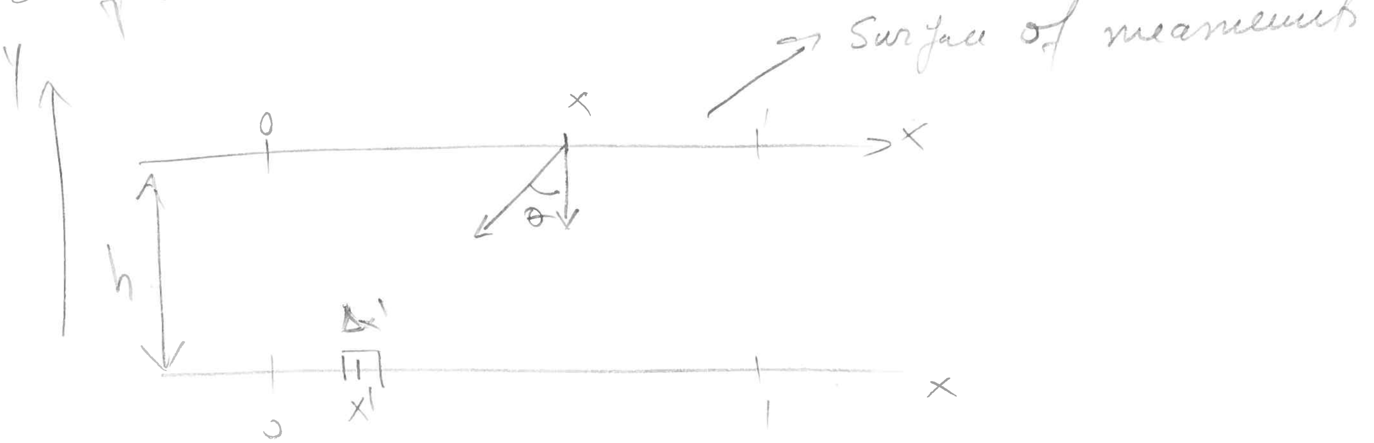
In fact,

$$u(x,T) = \frac{2}{\pi}\int_0^{\pi} k(x,y)\, u_0(y)\,dy$$

where $k(x,y) = \sum_{n=1}^{\infty} e^{-n^2 T} \sin(nx)\sin(ny).$

## Geological prospecting

Determine heterogeneities of Earth's interior from surface measurements.




Mass of volume element centered at $x'$, $\rho(x')\Delta x'$

Newton's law of gravity says (assuming mass at $x$ is 1).

$$\Delta f_y(x) = \underbrace{\Lambda}_{\text{gravitational constant.}} \frac{\rho(x')\Delta x'}{(x-x')^2+h^2}\cdot \cos\theta = \Lambda \frac{h\,\rho(x')\Delta x'}{\left[(x-x')^2+h^2\right]^{3/2}}$$

So the force with which the observer is pulled

$$f_y(x) = \Lambda h \int_0^1 \frac{\rho(x')}{\left((x-x')^2+h^2\right)^{3/2}}\,dx'$$

→ Measured change of gravity !! at the surface.

# Ill- posed problems vs. well posed problems

A problem is well posed if

1. There exists a solution.
2. There is at most a solution
3. The solution depends continuously on the data

$K : X \longrightarrow Y$ linear mapping.

$Kx = y$ – well posed if

$\exists! x$, $Kx = y$ and if for $x_n$, $Kx_n \to Kx \Rightarrow x_n \to x$

If any of the properties does not hold we have an ill posed problem.

Theorem.

Compact operators $Y, X$ - normed space

$K: X \to Y$ compact if it maps any bounded set in $X$ into a relatively compact set in $Y$.

$M \subset X$ is relatively compact if $\overline{M}$ is compact.
$U \subset X$ is compact if every sequence contains a subsequence convergent in $U$.

Theorem $K: X \to Y$ compact iff for each bounded sequence $\{\phi_n\}_n \subset X$, $K\phi_n$ contains a convergent subsequence.

Theorem Compact linear operators are bounded!

By contradiction Assume $K: X \to Y$ is not bounded
$\exists \{\phi_n\}_n \subset X$ s.t. $\|\phi_n\| = 1$ and $\|K\phi_n\| \geq n$ $\forall n \in \mathbb{N}$.
$\exists \phi_{n_j}$ s.t. $K\phi_{n_j} \to v \in Y$ (Compactness).
Contradiction!

**Theorem.** Any linear combination of compact operators is compact

**Theorem.** if $A : X \longrightarrow Y$ and $B : Y \longrightarrow Z$, bounded linear operators.

BA compact if A or B are compact.

**Theorem.** $K : X \longrightarrow Y$ is bounded linear operator. if $R(K)$ is finite dimensional then $K$ is compact.

— Ill-posed problems —

$K: X \longrightarrow Y$ compact, linear. Assume $K$ is one to one

Then $K^{-1}: Y \longrightarrow X$, if defined and is not bounded, if $X$ is infinite dimensional!

Otherwise $I = K^{-1}K : X \longrightarrow X$ is compact

$\Downarrow$

$X$ finite dimensional

Indeed.

1. Assume $X$ is not finite dimensional with $I: X \longrightarrow X$ compact.

Then let $\phi_1 \in X$ with $\|\phi_1\| = 1$. $U_1 = \text{span}\{\phi_1\}$ has dimension 1 and so $U_1 \subsetneq X$ (proper subspace)

Show now that for any $U \subsetneq X$, $U$ closed space there exists an element $\psi \in X$, $\|\psi\| = 1$ s.t.

$$\|\psi - \phi\| \geq \alpha \quad \text{for } \alpha \in (0,1).$$

for all $\phi \in U$

Since $U \neq X$, $\exists f \in X \setminus U$ and since $U$ is closed

$$\beta = \inf_{U} \|f - \phi\| > 0. \quad (1)$$

(Otherwise $\phi_n \in U$, $\phi_n \to f \to f \in U$)

Choose $v \in U$ s.t.

$$(2) \quad \beta \leq \|f - v\| \leq \frac{\beta}{\alpha} \quad (\alpha < 1).$$

Then $\psi = \dfrac{f - v}{\|f - v\|}$. So $\|\psi\| = 1$ and

$$\|\psi - \phi\| = \frac{1}{\|f - v\|} \|f - v - \phi\|f - v\|\| \underset{(1)}{\geq} \frac{\beta}{\|f - v\|} \underset{(2)}{\geq} \alpha.$$

So fr $\phi_1$ above and $U_1 = \text{span}\{\phi_1\}$, $\exists \phi_2 \in X$ $\|\phi_2\| = 1$ s.t $\|\phi_2 - \phi_1\| \geq \frac{1}{2}$ (Applying theorem with $\alpha = \frac{1}{2}$.)

$U_2 = \text{span}\{\phi_1, \phi_2\} \subsetneq X$ closed subspace

$\exists \phi_3$, $\|\phi_3\| = 1$ s.t.

$$\|\phi_3 - \phi_1\| \geq \tfrac{1}{2}, \|\phi_3 - \phi_2\| \geq \tfrac{1}{2}$$

Repeat and end up with $\{\phi_n\}_n \subset X$, $\|\phi_n\| = 1$ not Cauchy. Contradiction.

Because $K^{-1}$ is unbounded there exists a sequence $\{z_n\}_n \subset X$ s.t.

$$Kz_n \longrightarrow 0 \quad \text{and} \quad \|z_n\| \to \infty.$$

Indeed $K^{-1}$ unbounded means that there exists $v_n \in Y$, $\|v_n\| = 1$ s.t. $\|K^{-1}v_n\| \longrightarrow \infty$.

$$\frac{K^{-1}v_n}{\sqrt{\|K^{-1}v_n\|}} = z_n .$$

We see that $\|z_n\| = \sqrt{\|K^{-1}v_n\|} \longrightarrow \infty$.

but

$$Kz_n = \frac{v_n}{\sqrt{\|K^{-1}v_n\|}} \longrightarrow 0.$$

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Redo the proof when $K$ is not necessarily one to one

Consider now $X/_{N(K)}$ - factor space where

$N(K) = \{x \in X, Kx = 0\}$.

$X/_{N(K)}$ normed space with $\|x\| = \inf\{\|x+z\|, z \in N(K)\}$.

Assume now $X/_{N(K)}$ has infinite dimension

## Worst-Case Error

**Definition.** $K : X \longrightarrow Y$ linear bounded operator. $X_1 \subset X$ subspace such that $\|x\| \leq C\|x\|_1$ for some $C$, and for $x \in X_1$.

$$E(\delta, M, \|\cdot\|_1) = \sup\{\|x\|,\ x \in X_1,\ \|Kx\| \leq \delta,\ \|x\|_1 \leq M\}$$

$E(\delta, M, \|\cdot\|_1)$ - worst case error for the error $\delta$ in the data and apriori information

$$\|x\|_1 \leq M.$$

Ideally $E(\delta, M, \|\cdot\|_1) \xrightarrow[\delta]{} 0$ of order $\sigma(\delta)$. (3)

For the case when $K$ is invertible

$$\|x\| \leq \|K^{-1}\|\,\|Kx\|$$ and so we

have (3). !

Proposition $K : X \longrightarrow Y$ linear, one-to-one, compact. and assume $X$ is infinite dimensional. Then for every $M > 0$, there exists $c > 0$ and $\delta_0 > 0$ s.t. $E(\delta, M, \|\cdot\|) \geq c$ for all $\delta \in (0, \delta_0)$

Proof Assume $\exists\, \delta_n \longrightarrow 0$ s.t. $E(\delta_n, M, \|\cdot\|) \xrightarrow[n]{} 0$.

Let $K x_m \longrightarrow 0$. Then there exists a subsequence $x_{m_j}$ s.t. $\|K x_{m_j}\| \leq \delta_j$ for all $j \in \mathbb{N}$.

Let

$$z_j = \begin{cases} x_{m_j} & \text{if } \|x_{m_j}\| \leq M \\ M \cdot \dfrac{x_{m_j}}{\|x_{m_j}\|} & \text{if } \|x_{m_j}\| > M \end{cases}$$

Then $\|z_j\| \leq M$ and $\|K z_j\| \leq \delta_j$ for all $j \in \mathbb{N}$

From the hypothesis $E(\delta_j, M, \|\cdot\|) \xrightarrow[j]{} 0$ we have $\|z_j\| \longrightarrow 0$. Thus $z_j = x_{m_j}$ for large $j$!

Therefore $x_{m_j} \xrightarrow[j]{} 0$. Using this for all subsequences by Uryson theorem we have that $x_m \xrightarrow[m]{} 0$ and this implies continuity of $K^{-1}$ which contradicts the fact that $X$ is infinite dimensional

# Compact Self-Adjoint Operators

$X, Y$ - Hilbert spaces, $K: X \to Y$, bounded, linear

Adjoint $K^*: Y \to X$ defined by

$$(K\varphi, \psi) = (\varphi, K^*\psi). \quad \text{for all } \varphi \in X, \psi \in Y$$

**Theorem.** We have

$$(K(X))^\perp = N(K^*) \quad \text{and} \quad N(K^*)^\perp = \overline{K(X)}$$

**Proof.** $K(X)^\perp = N(K^*)$ — by definition

$U = K(X)$. Use $\overline{U} \subset (U^\perp)^\perp$.

Let $P: Y \to \overline{U}$ orthogonal projection

Let $\varphi \in ((U)^\perp)^\perp \Rightarrow P\varphi - \varphi \perp U$

But $P\varphi - \varphi \perp U^\perp$ because $\overline{U} \subset (U^\perp)^\perp$

Thus $P\varphi - \varphi = 0 \Rightarrow P\varphi = \varphi \Rightarrow \varphi \in \overline{U}$.

# Worst case error with apriori information

$X$, $Y$ - Hilbert spaces

$K : X \longrightarrow Y$ linear, compact and one-to-one

with $\overline{R(K)} = Y$

Let $K^*: Y \longrightarrow X$ define the adjoint operator

$$(K\varphi, \psi) = (\varphi, K^*\psi) \quad , \quad \forall \varphi \in X, \psi \in Y$$

a) if $X_1 = R(K^*)$, with norm

$$\|x\|_1 = \|(K^*)^{-1}x\|_Y \quad \text{for } x \in X_1.$$

Then.

$$E(\delta, M, \|\cdot\|_1) \leq \sqrt{\delta M}$$

The estimate is asymptotically sharp.

b). Let $X_2 = R(K^*K)$, $\|x\|_2 = \|(K^*K)^{-1}x\|_X$ for $x \in X_2$.

Then.

$$E(\delta, M, \|\cdot\|_2) \leq \delta^{2/3} M^{1/3}$$

The estimate is also asymptotically sharp.

Proof a) First note that

$$\|x\| \leq \|K^*\| \cdot \|x\|_1$$

Then ~~for~~ let $x = K^* z \in X_1$, $\|Kx\|_Y \le \delta$, $\|x\|_1 \le M$.

$$\|x\|_X^2 = (x, x) = (K^* z, x)_X = (z, Kx)_Y \le \|z\|_Y \cdot \|Kx\|_Y$$
$$\le M \cdot \delta.$$

where we used that $\|x\|_1 \le M \Rightarrow \|z\|_Y \le M$.

b). Note that:

$$\|x\|_X \le \|K\|^2 \cdot \|(K^*K)^{-1} x\|_X = \|K\|^2 \cdot \|x\|_2.$$

Let $x = K^*K z$, with $\|Kx\|_Y \le \delta$, $\|x\|_1 \le M$. $\Longrightarrow \|z\|_{\cancel{X}} \le M$.

$$\|x\|_X^2 = (x, x)_X = (K^*Kz, x) = (Kz, Kx)_Y$$
$$\le \|Kz\|_Y \cdot \|Kx\|_Y$$

$$\|Kz\|_Y^2 = (Kz \cdot Kz)_Y = (z, K^*Kz)$$
$$\le \|z\|_Y \cdot \|x\|_X$$

So
$$\|x\|_X^{\frac{3}{2}} \le \sqrt{\|z\|_Y} \cdot \|Kx\|_Y \le M^{1/2} \cdot \delta$$

$$\Downarrow$$

$$\|x\|_X \le M^{1/3} \delta^{2/3}.$$

# Solving linear systems

$$Ax = b$$

$$A \in M_{m \times n}(\mathbb{C}) \quad (m \geq n)$$

A vector $x \in \mathbb{C}^n$ minimizes the residual $\|b - Ax\|_2$ if and only if $(b - Ax) \perp R(A)$

Thus.

$$A^*(b - Ax) = 0 \,.$$

$A^*A$ is nonsingular if and only if $A$ is full rank

(In fact $x =$ proj $b$ on $R(A)$).

Remark. $A$ - full rank

$$A^+ = (A^*A)^{-1}A^* \quad \text{- pseudo inverse.}$$

In general, even if $A$ is not full rank, apply Tikhonov regularisation!

Solution given by

$$\hat{x} = (A^*A + \alpha I)^{-1} A^* b$$

SVD

There exists two unitary matrices $U$ $m \times m$ and $W$ $n \times n$ such that, rank $A = r$

$$U^* A W = \Sigma = \begin{array}{c} \\ \\ m-n\left\{ \right. \end{array} \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & \underbrace{0 \ \ddots \ 0}_{n-r} & \\ & & 0 & & & \end{bmatrix}_{m \times n}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0.$$

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$m \geq n \Rightarrow \sigma_i = \sqrt{\lambda_i}$, $i = 1, 2, \ldots, n$ where $\lambda_i$ are eigenvalues of $A^* A$

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Let $\Sigma^+ = \operatorname{diag}_{n \times m} \{ \frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_r}, \underbrace{0 \ldots 0}_{n-r} \}$

$$A^+ = W \Sigma^+ U^*$$ — Moore Penrose pseudo inverse of $A$.

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$\hat{x} = A^+ f$ minimizes $\| Ax - b \|_2$.

# Ill-posed problems

## General regularisation theory — $\boxed{Kx = y}$

$K : X \longrightarrow Y$, $K$ linear, compact one-to-one.

In practice we only know a noisy measurement; $y_\delta$

$$\| y - y_\delta \|_Y \le \delta \quad \text{for some } \delta > 0.$$

Want to solve

(1) $\quad K x_\delta = y_\delta$ — perturbed equation

(1) — not solvable in general because $y_\delta \notin R(K)$ in general

So we hope that the approximate $x_\delta$ is such that $K x_\delta \approx y$ and that $\| x_\delta - x_\eta \| \le E(\delta, M, \|\cdot\|, )$ when $K x_\eta \sim y$. for some $\eta > 0$.

Regularisation is a continuous approximation $R : Y \longrightarrow X$ of $K^{-1} : R(K) \longrightarrow X$

## Definition

A regularisation strategy is a family of linear and bounded operators

$$R_\alpha : Y \longrightarrow X \quad , \alpha > 0$$

such that

$$\lim_{\alpha \to 0} R_\alpha K x = x \quad \text{for all } x \in X$$

— $R_\alpha K$ converge pointwise to $I_X$.

**Theorem.** Let $R_\alpha$ be a regularisation strategy for a compact operator $K : X \longrightarrow Y$, with $\dim X = \infty$.

Then

i) $\{R_\alpha\}_\alpha$ are not uniformly bounded, i.e., $\exists \{\alpha_j\}_j$
$\|R_{\alpha_j}\| \longrightarrow \infty$ for $j \longrightarrow \infty$.

ii) $(R_\alpha K x)_\alpha$ does not converges uniformly on bounded sets of $X$, i.e. $R_\alpha K$ does not converge to $I$ in the operatorial norm

Proof.

i). By contradiction. assume $\exists\ c > 0$ s.t.

$$\|R_\alpha\| \leq c \text{ for all } \alpha > 0. \quad (2)$$

We have that (Def of regularisation).

$$R_\alpha Y \xrightarrow[\alpha]{} K^{-1}Y \quad \text{for all } Y \in R(K) \ (3).$$

(2) + (3). $\Rightarrow$ $\|K^{-1}Y\| \leq c\|Y\|$ for every $Y \in R(K)$

which implies $K^{-1}$ bounded. Contradiction with

$\dim X = \infty$.

ii) Assume, by contradiction, $R_\alpha K \longrightarrow I_X$ in the operatorial sense.

That implies compactness of $I_X$. Contradiction

with $\dim X = \infty$.

Remark $A_n : X \longrightarrow Y$ compact linear operator.

s.t. $\|A_n - A\| \xrightarrow[n]{} 0$ when $A: X \longrightarrow Y$. linear

Then $A$ is compact

Proof Hint. $\|\phi_m\| \leq M$. Show the exist subsequence

$\phi_{m_j}$ s.t. $A_n \phi_{m_j}$ converges for every fixed

$n$ as $j \to \infty$. (Diagonalisation).

Indeed by compactness.

$\exists \{\Phi_{m_1(j)}\}_j$ s.t $A_1 \Phi_{m_1(j)}$ convergent.

$\exists \{\Phi_{m_2(j)}\}$ subsequence of $\{\Phi_{m_1(j)}\}_j$ s.t

$A_2 \Phi_{m_2(j)}$ convergent.

$\exists \{\Phi_{m_n(j)}\}_j$ subsequence of $\{\Phi_{m_{n-1}(j)}\}_j$ s.t

$A \Phi_{m_n(j)}$ convergent

Define $\Phi_{m_j} = \Phi_{m_j(j)}$ and obtain the claim.

Then $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\| A_{n_0} - A \| < \varepsilon$$

$\exists\, N(\varepsilon)$ s.t.

$$\| A_{n_0} \Phi_{m_j} - A_{n_0} \Phi_{m(l)} \| < \varepsilon \quad , \quad j, l \geq N(\varepsilon)$$

$$\| A \Phi_{m_j} - A \Phi_{m_l} \| \leq \| A \Phi_{m_j} - A_{n_0} \Phi_{m_j} \| + \| A_{n_0} \Phi_{m_j} - A_{n_0} \Phi_{m_l} \| +$$
$$+ \| A_{n_0} \Phi_{m_l} - A \Phi_{m_l} \| < 3 \varepsilon \cdot C.$$

$\forall\, j, l \geq N(\varepsilon)$.

$A \Phi_{m_j}$ — Cauchy

Note that by definition

$$R_\alpha Y \longrightarrow x \quad \text{for} \quad Y = Kx.$$

So regularisation provide approximate solution to problem with unperturbed data.

For noisy data $Y_\delta \in \mathcal{Y}$, $Y_\delta \notin R(K)$

$$\| Y_\delta - Y \| \leq \delta.$$

Consider the solution by regularisation.

$$x^{\alpha,\delta} := R_\alpha Y^\delta$$

$$\| x^{\alpha,\delta} - x \| \leq \| R_\alpha Y^\delta - R_\alpha Y \| + \| R_\alpha Y - x \|$$

$$\leq \| R_\alpha \| \, \| Y^\delta - Y \| + \| R_\alpha K x - x \|.$$

$$\| x^{\alpha,\delta} - x \| \leq \underbrace{\delta \| R_\alpha \|}_{\substack{\text{data error} \\ \xrightarrow[\alpha]{} \infty}} + \underbrace{\| R_\alpha K x - x \|}_{\substack{\text{approximation error} \\ \xrightarrow[\alpha]{} 0}}$$

Need strategy to choose $\alpha = \alpha(\delta)$ so that the error is small.

$A: X \longrightarrow X \qquad A = A^* \;-\;$ self adjoint.

Thm. Bounded self adjoint operator.

$$\|A\| = \sup_{\|\varphi\|=1} |(A\varphi, \varphi)|.$$

Proof. Recall that

$$\|A\| = \sup_{\substack{\varphi \in X \\ \varphi \neq 0}} \frac{\|A\varphi\|}{\|\varphi\|}$$

$(A\varphi, \varphi) \leq \|A\|$ for all $\|\varphi\|=1$.

Apply sup. and get.

$$\sup_{\|\varphi\|=1} (A\varphi, \varphi) \leq \|A\|$$

Note that

$$(A(\varphi+\psi), \varphi+\psi) - (A(\varphi-\psi), \varphi-\psi) = 2\left((A\varphi, \psi) + (A\psi, \varphi)\right) = 4 \; (A\varphi, \psi)$$

$$4 \; (A\varphi, \psi) \leq \sup_{\|\varphi\|=1} (A\varphi, \varphi) \cdot \{\|\varphi+\psi\|^2 + \|\varphi-\psi\|^2\} =$$

$$= 2\, q \{\|\varphi\|^2 + \|\psi\|^2\}.$$

Let $\|\varphi\|=1$, $A\varphi \neq 0$. Choose $\psi = \|A\varphi\|^{-1} \cdot A\varphi$

$$\|A\varphi\| \leq (A\varphi, \psi) \leq q.$$

**Thm** All eigenvalues of a self-adjoint operator are real and eigenelements to different eigenvalues are orthogonal.

**Thm**. The spectral radius of a bounded self-adjoint operator $A$ satisfies

$$r(A) = \|A\|.$$

If $A$ is compact, then there exists at least one eigenvalue $|\lambda| = \|A\|$.

Note $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$ , $\sigma(A)$ - spectrum of $A$.
Complement of $\rho(A)$
$\rho(A) = \{\lambda \text{ s.t. } (\lambda I - A)^{-1} \text{ exists and is bounded}\}$

Proof $r(A) \leq \|A\|$ because otherwise

fix $\lambda \in \mathbb{C}$, $|\lambda| > \|A\|$ then

$|\lambda| I - A$ is invertible by Neuman Series.

$$(|\lambda| I - A)^{-1} = |\lambda|^{-1} \left(I - \frac{A}{|\lambda|}\right)^{-1} = |\lambda|^{-1} \sum_{k=0}^{\infty} A^k$$

So if $\lambda \in \mathbb{C}$, $|\lambda| > \|A\| \Rightarrow \lambda \notin \sigma(A)$

$\exists \{\varphi_n\}_{n=1}^{\infty}$ s.t. $\|\varphi_n\| = 1$ such that

$$|(A\varphi, \varphi)| \longrightarrow \|A\| \ , \ n \longrightarrow \infty.$$

$$0 \le \| A\varphi_n - \|A\|\varphi_n\|^2 = \|A\varphi_n\|^2 - 2\|A\|(A\varphi_n, \varphi_n) + \|A\|^2 \cdot \|\varphi_n\|^2$$

$$\le \|A\|^2 - 2\|A\|(A\varphi_n, \varphi_n) + \|A\|^2 = 2\|A\|\{\|A\| - (A\varphi_n, \varphi_n)\} \xrightarrow[n]{} 0$$

So $\quad A\varphi_n - \|A\|\varphi_n \xrightarrow[n]{} 0 \qquad (*)$

So $\lambda = \|A\|$ is in $\sigma(A)$ otherwise if not,

$$1 = \|\varphi_n\| = \|(\lambda I - A)^{-1}(\lambda\varphi_n - A\varphi_n)\| \xrightarrow[n]{} 0$$

So $r(A) \ge |\lambda| = \|A\|$.

If $A$ compact then there exists $\varphi_{n_j}$ st

$A\varphi_{n_j} \longrightarrow \psi$. This implies with $(*)$

$\varphi_{n_j}$ convergent, $\varphi_{n_j} \longrightarrow \varphi$

$\|\varphi_{n_j}\| = 1 \Rightarrow \|\varphi\| = 1$. And continuity of $A$ implies

$A\varphi_{n_j} \longrightarrow A\varphi \qquad (**)$

$(*)$, $(**)$ imply $\lambda = \|A\|$ is an eigenvalue!

Note that in general $\sigma(A)$ is not point spectrum!

Thm. $A : X \to X$ compact linear operator on $X$, $\dim X = +\infty$. Then

1. $\lambda = 0$ is in the spectrum
2. $\sigma(A) \setminus \{0\}$ consists of at most countable set of eigenvalues with no point of accumulation except $\lambda = 0$.

1. Otherwise $A^{-1}$ exists and is bounded. Contradiction!

2. Riesz theory. - See reference books!

Theorem - Spectral theorem !!

$A : X \to X$, $A = A^*$, $X$ - Hilbert space, $A \neq 0$.
All eigenvalues are real. $A$ has at least one eigenvalue different from zero and at most a countable set of eigenvalues accumulating only at zero.
$N(\lambda I - A)$ have finite dimension for $\lambda \neq 0$ and
$N(\lambda I - A) \perp N(\mu I - A)$ for $\lambda \neq \mu$.

Assume $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and denote

$P_j : X \longrightarrow N_j(\lambda I - A)$ the orthogonal projection operator. Then

$$A = \sum_{j=1}^{\infty} \lambda_j P_j \quad \text{in the sense of}$$

norm convergence. Let $Q : X \longrightarrow N(A)$ denote the orthogonal projection on $N(A)$. Then

$$x = \sum_{j=1}^{\infty} P_j x + Qx \quad \text{for all } x \in X.$$

<u>Note</u> In fact if $\{x_j\}_j$ is a sequence of orthonormal eigenelements then

$$x = \sum_{j=1}^{\infty} (x, x_j) x_j + Qx$$

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, x_j) x_j.$$

where we repeat $\lambda_j$'s according to their multiplicity

# Singular Value Decomposition

<u>Definition</u>. $X, Y$ — Hilbert spaces.

$K: X \longrightarrow Y$ compact linear operator

$K^*: Y \longrightarrow X$ adjoint.

The nonnegative square roots of eigenvalues of $K^*K: X \longrightarrow X$ are called singular values of $A$

<u>Thm</u>. $\{\mu_j\}_j$ — sequence of nonzero singular values of the compact linear operator $K$. $(A \neq 0)$ repeated according to their multiplicity. Then there exist orthonormal sequences $(x_j)_j$ in $X$ and $\{y_j\}_j$ in $Y$ such that $\forall j \in \mathbb{N}$.

$$K x_j = \mu_j y_j, \quad K^* y_j = \mu_j x_j$$

$$x = \sum_{j=1}^{\infty} (x, x_j) x_j + Qx \quad \text{for all } x \in X$$

where $Q: X \longrightarrow N(K)$ orthogonal proj and

$$Kx = \sum_{j=1}^{\infty} \mu_j (x, x_j) y_j .$$

$(\mu_j, x_j, y_j)$ singular system of $K$

Proof

$\{x_j\}_j$ orthonormal system of eigenfunctions corresponding to $K^*K$ for $\{\mu_j^2\}_j$

$$K^*K\, x_j = \mu_j^2\, x_j$$

Define
$$y_j = \frac{1}{\mu_j} K x_j \quad \text{— orthonormal.}$$

Apply the spectral decomposition theorem to $K^*K$

$$\varphi = \sum_{n=1}^{\infty} (\varphi, x_j)\, x_j + Q\varphi \quad , \; \varphi \in X.$$

$$Q : X \longrightarrow N(K^*K).$$

Show $N(K^*K) = N(K)$

# Picard Theorem

$K : X \to Y$ compact linear operator.
with singular system $(\mu_j, x_j, y_j)$.

$Kx = y$ is solvable iff

$y \in N(K^*)^\perp$ and

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j^2} |(y, y_j)|^2 < \infty \qquad (\square)$$

Then the solution is given by

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (y, y_j)\, x_j \qquad (\square\square)$$

**Proof** Necessity

Since $y = Kx \Rightarrow y \in R(K) \Rightarrow y \perp N(K^*)$

$$\frac{1}{\mu_j}(y, y_j) = \frac{1}{\mu_j}(Kx, y_j) = \frac{1}{\mu_j}(x, K^* y_j) = (x, x_j)$$

So $$\sum_{j=1}^{\infty} \frac{1}{\mu_j^2} |(y, y_j)|^2 < \sum_{j=1}^{\infty} |(x, x_j)|^2 \le \|x\|^2$$

Sufficiency. Note that $\sum s_n$ converges in $X$
if $\sum \|s_n\|$ converges in $\mathbb{R}$

Assume $Y \perp N(K^*)$ and (D) holds.

Then $\sum_{j=1}^{\infty} \frac{1}{\mu_j^2} |(Y, Y_j)|^2 < \infty$ implies

convergence of

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j} (Y, Y_j) X_j \quad \text{in } X.$$

Let

$$X = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (Y, Y_j) X_j .$$

Then

$$KX = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (Y, Y_j) \mu_j Y_j =$$

$$= \sum_{j=1}^{\infty} (Y, Y_j) Y_j = Y$$

follows from the fact

that $(\mu_j, Y_j, X_j)$ is a singular system for $K^*$ singular value decomposition for $K^*$

and using the

and $Y \perp N(K^*)$ we have

$$Y = \sum_{j=1}^{\infty} (Y, Y_j) Y_j$$

# Construction of admissible regularisation strategies.

## Definition

Regularisation strategy $R_\alpha$, $\alpha = \alpha(\delta)$ is called admissible if $\alpha(\delta) \to 0$ and

$$\sup\{\|R_{\alpha(\delta)} y^\delta - x\| ; y^\delta \in Y, \|Kx - y^\delta\| \le \delta\} \to 0$$

as $\delta \to 0$.

## Admissible Regularisation strategies by filtering singular systems

$K : X \to Y$ linear compact operator with $(\mu_j, x_j, y_j)$ singular system for $K$.

Solution of $Kx = y$ given by Picard theorem.

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j}(y, y_j)\, x_j$$

if the series converges. $y \in R(K)$

Note that if we perturb $Y$ by $Y_\delta = Y + \delta Y_j$.

We obtain $X_\delta = X + \delta \mu_j^{-1} x_j$

So $\dfrac{\|X_\delta - X\|}{\|Y_\delta - Y\|} = \dfrac{1}{\mu_j} \longrightarrow \infty$ because $\mu_j \longrightarrow 0$

Problem is mildly ill posed if $\mu_j \longrightarrow 0$ slowly and is severely ill-posed if $\mu_j \longrightarrow 0$ fast!

Thus construct Regularization strategy by dumping $\frac{1}{\mu_j}$

<u>Idea</u>: $R_\alpha Y = \sum_{j=1}^{\infty} \dfrac{q(\alpha, \mu_j)}{\mu_j} (Y, Y_j) x_j \quad , Y \in \mathcal{Y}$

with $q : (0, \infty) \times (0, \|K\|] \longrightarrow \mathbb{R}$

Specially designed

<u>Def</u>. Reg. strategy is admissable if $\alpha(\delta) \longrightarrow 0$ and

$\sup \{ \|R_{\alpha(\delta)} Y_\delta - Y\|, \; Y_\delta \in \mathcal{Y}, \|Kx - Y_\delta\| \le \delta \} \longrightarrow 0$

for every $x$.