Construction of admissible regularization strategies.

Definition

Regularization strategy R_α, $\alpha = \alpha(\delta)$ is called admissible if $\alpha(\delta) \to 0$ and

$$\sup \{ \| R_{\alpha(\delta)} y^\delta - x \| ; \; y^\delta \in Y, \; \| Kx - y^\delta \| \le \delta \} \to 0$$

as $\delta \to 0$.

Admissible Regularization strategies by filtering singular systems

$K : X \to Y$ linear compact operator.

with (μ_j, x_j, y_j) singular system for K

Solution of $Kx = y$ given by Picard theorem.

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (y, y_j) x_j$$ if the series converges.

$y \in R(K)$

Note that if we perturb Y by $Y_\delta = Y + \delta Y_j$.

We obtain $X_\delta = X + \delta \mu_j^{-1} x_j$

So $\dfrac{\|X_\delta - X\|}{\|Y_\delta - Y\|} = \dfrac{1}{\mu_j} \longrightarrow \infty$ because $\mu_j \longrightarrow 0$

Problem is mildly ill posed if $\mu_j \longrightarrow 0$ slowly

and is severely ill-posed if $\mu_j \longrightarrow 0$ fast!

Thus construct Regularization strategy by

dumping $\frac{1}{\mu_j}$

Idea: $R_\alpha Y = \sum_{j=1}^{\infty} \dfrac{q(\alpha, \mu_j)}{\mu_j} (Y, Y_j) x_j \quad , Y \in Y$

with $q : (0, \infty) \times (0, \|k\|] \longrightarrow \mathbb{R}$

Specially designed

Def Reg. strategy is admissible if $\alpha(\delta) \longrightarrow 0$

and

$\sup \{ \|R_{\alpha(\delta)} Y_\delta - Y\|, \; Y_\delta \in Y, \|kx - Y_\delta\| \le \delta \} \longrightarrow 0$

for every x.

Theorem 1

$K : X \to Y$ be compact, one to one, with singular system (μ_j, x_j, y_j) and

$$q : (0, \infty) \times (0, \|K\|] \longrightarrow \mathbb{R}$$

be a function with the properties.

(1) $|q(\alpha, \mu)| \leq 1$ for all $\alpha > 0$, $0 < \mu \leq \|K\|$

(2) For every $\alpha > 0$, there exists $c(\alpha)$ such that $|q(\alpha, \mu)| \leq c(\alpha)\mu$ for all $0 < \mu \leq \|K\|$

(3) $\lim_{\alpha \to 0} q(\alpha, \mu) = 1$ for every $0 < \mu \leq \|K\|$.

Then $R_\alpha : Y \to X$ defined by.

$$R_\alpha y = \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j} (y, y_j)\, x_j \quad , y \in Y$$

is a regularisation strategy. with $\|R_\alpha\| \leq c(\alpha)$. Choice $\alpha = \alpha(\delta)$ admissable if $\alpha(\delta) \to 0$ and

$\delta\, c(\alpha(\delta)) \to 0$ as $\delta \to 0$.

q - regularisation filter for K

Proof

$$\|R_\alpha y\|^2 = \sum_{j=1}^{\infty} |q(\alpha, \mu_j)|^2 \frac{1}{\mu_j^2} |(y, y_j)|^2 \le$$

$$\le c(\alpha)^2 \cdot \|y\|^2$$

So $\|R_\alpha\| \le c(\alpha)$

$$R_\alpha K x = \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j} (Kx, y_j) x_j \qquad (4)$$

$$x = \sum (x, x_j) x_j \qquad (5)$$

$$(Kx, y_j) = \mu_j (x, x_j) \qquad (6).$$

(6) in (4) together with (5) gives

$$\|R_\alpha K x - x\|^2 = \sum_{j=1}^{\infty} (q(\alpha, \mu_j) - 1)^2 |(x, x_j)|^2$$

Since $x \in X$, for $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$$\sum_{j=N+1}^{\infty} |(x, x_j)|^2 < \varepsilon$$

From (3), $\exists\ \alpha_0 > 0$ s.t

$$(g(\alpha, \mu_j) - 1)^2 < \varepsilon. \quad \text{for all } j = 1, \dots, N, \quad 0 < \alpha \le \alpha_0$$

Thus

$$\|R_\alpha K x - x\|^2 = \sum_{j=1}^{N} (g(\alpha, \mu_j) - 1)^2 |(x, x_j)|^2 +$$

$$+ \sum_{j=N+1}^{\infty} (g(\alpha, \mu_j) - 1)^2 |(x, x_j)|^2 <$$

$$< N \varepsilon \|x\| + 4 \cdot \varepsilon.$$

where in the last estimate we used (1). as well.

Optimal regularization strategies

Regularization strategies that converge with the same order as worst case error.!

$$\|R_\alpha y^\delta - x\| \le \underbrace{\delta \|R_\alpha\|} + \|R_\alpha K x - x\|$$

$$\le \delta c(\alpha) + \|R_\alpha K x - x\|$$

If $\alpha(\delta) \to 0$ with $\delta c(\alpha) \to 0$ then we have admissability!

Theorem 2

Let (1), (2) hold as in the Theorem 1

Let (3) be replaced by

(3 *) There exist $c_1 > 0$ with.

$$|g(\alpha, y) - 1| \le c_1 \frac{\sqrt{\alpha}}{y} \quad \forall \alpha > 0, \; 0 < y \le \|K\|$$

If $x \in R(K^*)$ then

$$\|R_\alpha K x - x\| \le c_1 \sqrt{\alpha} \, \|z\|.$$

when $x = K^* z$

if (3) replaced by, $\exists \, c_2 > 0$

$$|g(\alpha, y) - 1| \le c_2 \frac{\alpha}{y^2}, \quad \forall \alpha > 0, \; 0 < y \le \|K\|$$

and if $x = K^* K z$ then

$$\|R_\alpha K x - x\| \le c_2 \alpha \|z\|$$

Proof Idea $x = K^* z \Rightarrow (x, x_j) = y_j (z, y_j)$

$$\|R_\alpha K x - x\|^2 = \sum_{j=1}^{\infty} (g(\alpha, y_j) - 1)^2 y_j^2 |(z, y_j)|^2 \le c_1^2 \alpha \|z\|^2.$$

Theorems q_i, $i=1,2,3$ satisfy the assumption in Theorem 1, Theorem 2.

a) $q_1(\alpha, \mu) = \dfrac{\mu^2}{\alpha + \mu^2}$

b) $q_2(\alpha, \mu) = 1 - (1 - a\mu^2)^{1/\alpha}$ for some

$\alpha \in (0,1)$ $\quad 0 < a < \dfrac{1}{\|K\|^2}$ $\quad$ (Bernoulli's inequality $(1+x)^{n} \geq 1 + nx$ $\begin{cases} n \leq 0 \\ n \geq 1 \end{cases}$, $x > -1$

c) $q_3(\alpha, \mu) = \begin{cases} 1 & \mu^2 \geq \alpha \\ 0 & \mu^2 < \alpha \end{cases}$

Theorem $y_\delta \in Y$, $\|y_\delta - y\| \leq \delta$ where $y = Kx$

$K: X \to Y$ compact, one to one, with singular system (μ_j, x_j, y_j)

$$R_\alpha y = \sum_{\mu_j^2 \geq \alpha} \frac{1}{\mu_j} (y, y_j) x_j, \quad y \in Y$$

define a regularization strategy with $\|R_\alpha\| \leq \dfrac{1}{\sqrt{\alpha}}$

The strategy is admissible if $\alpha(\delta) \to 0$ and $\dfrac{\delta^2}{\alpha(\delta)} \to 0$

If $x = K^* z$ with $\|z\| \le M$ and $c > 0$ then.

For $\alpha(\delta) = c\dfrac{\delta}{M}$ we have

$$\| x_{\alpha,\delta} - x \| \le \left(\frac{1}{\sqrt{c}} + \sqrt{c} \right) \sqrt{\delta M}.$$

If $x = K^* K$ with $\|z\| \le M$, $c > 0$ for

$\alpha(\delta) = \dfrac{c\,\delta^{2/3}}{M^{2/3}}$ we have

$$\| x_{\alpha,\delta} - x \| \le \left(\frac{1}{\sqrt{c}} + c \right) M^{1/3} \delta^{2/3}.$$

The spectral cutoff strategy is optimally for $\|(K^*)^{-1} x\| \le M$ or the extra apriori information

$$\|(K^* K)^{-1} x\| \le M.$$

Tikhonov Regularization

(1) $K: X \longrightarrow Y$ linear and bounded and $y \in Y$.

$\exists\ \hat{x} \in X$ with $\|K\hat{x} - y\| \leq \|Kx - y\|$ for all $x \in X$.

if and only if $\hat{x} \in X$ solves.

$$K^* K \hat{x} = K^* y$$

$\hookrightarrow$ Normal equation!

Remark. Short parenthesis about the linear problem in $\mathbb{R}^{m \times n}$

$m > n$, $A \in \mathbb{C}^{m \times n}$, $b \in \mathbb{C}^m$

Find $x \in \mathbb{C}^n$ s.t $Ax = b$

If $m > n$, $b \notin R(A)$ in general.

Project b on $R(A)$. to minimize $b - Ax$.

One way to see the equation satisfied by projection

$$(b - A\hat{x}, Az) = 0 \quad \text{for } z \in \mathbb{C}^n.$$

$$\Rightarrow A^* b = A^* A \hat{x}.$$

Different approach

$$J(x) = \langle b - Ax, b - Ax \rangle$$

$J(\hat{x}) = \min_{x \in \mathbb{C}^n} J$, Compute differential of J.

$$J(\hat{x} + \delta y) - J(\hat{x}) = (b - A\hat{x} - A\delta y, b - A\hat{x} - A\delta y) -$$

$y \in \mathbb{C}^n$,

$$- (b - A\hat{x}, b - A\hat{x}) =$$

$$= \delta^2 (Ay, Ay) - 2\delta (b - A\hat{x}, Ay).$$

$\delta_x = y.$

$$\frac{J(\hat{x} + \delta y) - J(x)}{\delta} = \delta (Ay, Ay) - 2 (b - A\hat{x}, Ay)$$

$$\lim_{\delta \to 0} \frac{J(\hat{x} + \delta y) - J(\hat{x})}{\delta} = 0 \Rightarrow (b - A\hat{x}, Ay) = 0$$

$$\Rightarrow \boxed{A^* b = A^* A \hat{x}}$$

Proof of (1)

I Projection on the $R(k)$. ?

if $R(k)$ is not closed it may not exists!

Anyway, if it exists is characterized by.

$$(Y - k\hat{x}, kz) = 0 \quad \text{for all } z \in X$$

$$\Downarrow$$

$$k^* Y - k^* k \hat{x} = 0.$$

Note that $R(k)$ closed $\Leftrightarrow$ $K^{-1} : k(X) \longrightarrow X$ bounded

If $k^* y = k^* k \hat{x}$ then from.

$$\|kx - y\|^2 - \|k\hat{x} - y\|^2 = (kx - y, kx - y) - (k\hat{x} - y, k\hat{x} - y)$$

$$= \|kx - k\hat{x}\|^2 + (k\hat{x} - y, k(x - \hat{x})) +$$

$$+ (k(x - \hat{x}), k\hat{x} - y)$$

$$= \|kx - k\hat{x}\|^2 + 2(k\hat{x} - y, k(x - \hat{x}))$$

$$= \|kx - k\hat{x}\|^2 + 2(k^* k\hat{x} - k^* y, (x - \hat{x}))$$

In general $K: X \to Y$ is not s.t. $R(K)$ closed.

So. $K^*K\hat{x} = K^*y$ does not have a solution!!

Penalise the defect. by considering.

x_α solution of.

$$K^*K x_\alpha + \alpha x_\alpha = K^* y$$

x_α will minimize the Tikhonov functional

$$J_\alpha(x) = \|Kx - y\|^2 + \alpha\|x\|^2 \quad \text{for } x \in X$$

<u>Theorem</u>! $K: X \to Y$, linear, bounded, $\alpha > 0$.

Then J_α has a unique minimum $x_\alpha \in X$, which is the unique solution of

$$\alpha x_\alpha + K^*K x_\alpha = K^* y.$$

<u>Proof</u> Previous lemma.

$$\|Kx - y\|^2 + \alpha\|x\|^2 - \|Kx_\alpha - y\|^2 - \alpha\|x_\alpha\|^2 =$$

$$+ 2(x - x_\alpha, \alpha x_\alpha + K^*(Kx_\alpha - y)) + \qquad (*)$$

$$+ \|K(x - x_\alpha)\|^2 + \alpha\|x - x_\alpha\|^2$$

Next show that $\alpha x_\alpha + K^*K x_\alpha = K^*y$ has a unique solution!

$T: X \to X$, $T = \alpha I + K^* K$. (2)

$(Tx, x) \geq \alpha \|x\|^2$ — <u>strictly</u> <u>coercive</u>.

<u>Lax - Milgram Lemma</u>

$A: X \to X$ – linear and strictly coercive.
– bounded

Then $A^{-1}: X \to X$ is bounded.

Thus, $T^{-1}: X \to X$ defined at (2) is bounded!

Second solution for Theorem! step (*) as before. Now we show that the functional attains its infimum.

$I = \inf_X J_\alpha(x)$ exists. Let $\{x_n\}_n$ minimizing sequence, $J_\alpha(x_n) \xrightarrow[n]{} I$. Show that $\{x_n\}_n$ is Cauchy.

$$J_\alpha(x_m) + J_\alpha(x_n) = 2 J_\alpha\left(\tfrac{1}{2}(x_n + x_m)\right) + \tfrac{1}{2}\|K(x_n - x_m)\|^2 +$$

$$\downarrow_{m,n \to \infty} \qquad + \tfrac{\alpha}{2}\|x_n - x_m\|^2. \qquad (3)$$

$$2I$$

$$\geq 2I + \tfrac{\alpha}{2}\|x_m - x_n\|^2$$

Thus. $\|x_m - x_n\|$ can be made arbitrarily small for m, n large.

Let $x_\alpha = \lim_n x_n$, $x_\alpha \in X$.

J_α - continuous, $J_\alpha(x_n) \longrightarrow J_\alpha(x_\alpha) = I$.

For (3) we used binomial formula!

$$\|a\|^2 + \|b\|^2 = 2\left\|\frac{a+b}{2}\right\|^2 + 2\left\|\frac{a-b}{2}\right\|^2$$

---

x_α can be written as. $x_\alpha = R_\alpha y$ with

$R_\alpha : Y \to X$; $R_\alpha y = (\alpha I + K^*K)^{-1}K^*$

If (μ_j, x_j, y_j) is a singular system we see that.

$$R_\alpha y = \sum_{n=0}^{\infty} \frac{\mu_j}{\alpha + \mu_j^2}(y, y_j)\, x_j$$

$$q(\alpha, \mu_j)/\mu_j, \quad q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2}$$

Theorem 2. $K : X \to Y$, linear, compact, $\alpha > 0$.

i $(\alpha I + K^* K)^{-1}$, bounded.

$$R_\alpha = (\alpha I + K^* K) K^* : Y \to X \quad \text{form a}$$

regularization strategy. with $\|R_\alpha\| \le \frac{1}{2\sqrt{\alpha}}$

$R_\alpha Y_\delta$ is the unique solution of

$$\alpha x_{\alpha,\delta} + K^* K x_{\alpha,\delta} = K^* Y_\delta.$$

$\alpha(\delta) \xrightarrow[\delta]{} 0$ with $\frac{\delta^2}{\alpha(\delta)} \longrightarrow 0$ is admissible

b). if $x = K^* z \in R(K^*)$ with $\|z\| \le M$ then

for $\alpha(\delta) = \frac{c\delta}{M}$, $c > 0$, we have

$$\|x_{\alpha,\delta} - x\| \le \frac{1}{2}\left(\frac{1}{\sqrt{c}} + \sqrt{c}\right)\sqrt{\delta M}$$

c) $x = K^* K z$ with $\|z\| \le M$. Then $\alpha(\delta) = c\left(\frac{\delta}{M}\right)^{2/3}$.

$c > 0$, gives.

$$\|x_{\alpha,\delta} - x\| \le \left(\frac{1}{2\sqrt{c}} + c\right) M^{1/3} \delta^{2/3}.$$

Tikhonov regularization is optimal for $\|(K^*)^{-1} x\| \le M$ or

$\|(K^* K)^{-1} x\| \le M$.

Theorem $K: X \to Y$ linear, compact, one-to-one
$R(K)$ infinite dimensional.

Assume there exist $\alpha: [0,\infty) \to [0,\infty)$ $\alpha(0)=0$ s.t.

$\lim_{\delta} \| x_{\alpha,\delta} - x \| \delta^{-2/3} = 0.$ for every $\| y_\delta - Kx \| \leq \delta$

where $x_{\alpha,\delta} \in X$ solution of.

$$\alpha x_{\alpha,\delta} + K^* K x_{\alpha,\delta} = K^* y_\delta.$$

Then $x = 0$

Proof Assume $x \neq 0$.

Show that $\alpha \delta^{-2/3} \to 0$. $y = Kx$. Then.

$$(\alpha I + K^* K)(x_{\alpha,\delta} - x) = K^*(y_\delta - y) - \alpha x.$$

$$|\alpha| \|x\| \leq \|K\| \delta + (\alpha + \|K\|^2) \| x_{\alpha,\delta} - x \|$$

$$\cancel{\alpha \cdot \delta^{-2/3} \|x\| \leq \|K\|^{1/3} (\alpha + \|K\|^2)}$$

$$\alpha \cdot (\|x\| - \|x_{\alpha,\delta} - x\|) \leq \|K\| \delta + \|K\|^2 \|x_{\alpha,\delta} - x\|$$

$$\alpha \cdot \delta^{-2/3} \to 0$$

(μ_j, x_j, y_j) singular system for K.

$$\delta_j = \mu_j^3, \quad y_{\delta_j} = y + \delta_j y_j \quad , j \in \mathbb{N}.$$

$$\alpha_j = \alpha(\delta_j)$$

$$x_{\alpha_j, \delta_j} - x = (x_{\alpha_j, \delta_j} - x_{\alpha_j}) + (x_{\alpha_j} - x) =$$

$$= (\alpha_j I + K^* K)^{-1} K^* (\delta_j y_j) + (x_{\alpha_j} - x)$$

$$= \frac{\delta_j \mu_j}{\alpha_j + \mu_j^2} x_j + x_{\alpha_j} - x$$

Since $\|x_{\alpha_j} - x\| \delta_j^{-2/3} \to 0$

we have.

$$\frac{\delta_j^{1/3} \mu_j}{\alpha_j + \mu_j^2} \to 0 \quad j \to \infty.$$

Contr.

But. $$\frac{\delta_j^{1/3} \mu_j}{\alpha_j + \mu_j^2} = \frac{\mu_j^2}{\alpha_j + \mu_j^2} = (1 + \alpha_j \delta_j^{-2/3})^{-1} \to 1$$

Discrepancy Principle of Morozov

A posteriori choice of regularization parameter α!

$K : X \to Y$ - compact and injective

$\overline{K(X)} = Y$

$K x = y$ for $y \in Y$. (May not have a sol. if $y \notin R(K)$).

With Tikhonov regularization.

Let $x_\alpha = (\alpha I + K^* K)^{-1} K^*$ be the unique solution

of $K^* K x_\alpha + \alpha x_\alpha = K^* y$

1. x_α depends continuously on α.

2. $\alpha \to \|x_\alpha\|$ monotoniously nonincreasing

3. $\alpha \longrightarrow \|K x_\alpha - y\|$ monotoniously nondecreasing.

and $\lim_{\alpha \to 0} K x_\alpha = y$.

If $y \notin N(K^*)$ then strict monotonicity is observed in 2 and 3.

Proof

Remember x_α is minimizer of.

$$J_\alpha(x) = \|Kx - y\|^2 + \alpha\|x\|^2 \quad \text{on } X$$

Thus.

$$\alpha\|x_\alpha\| \leq J_\alpha(x_\alpha) \leq J_\alpha(0) = \|y\|^2.$$

$$\Downarrow$$

$$\|x_\alpha\| \leq \frac{\|y\|}{\sqrt{\alpha}} \Rightarrow x_\alpha \to 0 \text{ as } \alpha \to \infty$$

Let $\alpha, \beta > 0$, let $\varepsilon > 0$.

$$\alpha(x_\alpha - x_\beta) + K^*K(x_\alpha - x_\beta) + (\alpha - \beta)x_\beta = 0$$

$$(1)\quad \alpha\|x_\alpha - x_\beta\|^2 + \|K(x_\alpha - x_\beta)\|^2 = (\beta - \alpha)(x_\beta, x_\alpha - x_\beta)$$

$$\Downarrow$$

$$\alpha\|x_\alpha - x_\beta\|^2 \leq |\beta - \alpha|\,\|x_\beta\|\,\|x_\alpha - x_\beta\|$$

$$\alpha\|x_\beta - x_\alpha\| \leq |\beta - \alpha|\frac{\|y\|}{\sqrt{\beta}} < \varepsilon\frac{\|y\|}{\sqrt{\beta}}$$

Choose $\delta = \dfrac{\varepsilon\|y\|}{\alpha\cdot\sqrt{\beta}}$ and see continuity.

Then let $\beta > \alpha > 0$. Then by using (1)

we obtain

$$(x_\beta, x_\alpha - x_\beta) \geq 0. \quad \text{So, we have.}$$

$$\|x_\beta\|^2 \leq (x_\beta, x_\alpha) \leq \|x_\beta\| \|x_\alpha\|$$

$$\Downarrow$$

$$\|x_\beta\| \leq \|x_\alpha\|$$

$\hookrightarrow \alpha — \|x_\alpha\|$ - monotone nonincreasing

Next;

multiply

$$\beta x_\beta + K^* K x_\beta = K^* Y$$

by $(x_\alpha - x_\beta)$ to obtain.

$$\beta (x_\beta, x_\alpha - x_\beta) + (K x_\beta - Y, K(x_\alpha - x_\beta)) = 0 \quad (2)$$

Let $\beta > \alpha$. From (1) as before.

$$(x_\beta, x_\alpha - x_\beta) \geq 0$$

From this and (2) we have.

$$(K x_\beta - Y, K(x_\alpha - x_\beta)) \leq 0 \Longleftrightarrow$$

$$(K x_\beta - Y, K x_\alpha - Y) \leq \|K x_\beta - Y\|^2$$

$\|Kx_\alpha - Y\| \le \|Kx_\beta - Y\|$

$\alpha \longmapsto \|Kx_\alpha - Y\|$ - monotonous nondecreasing.

Next, let $\varepsilon > 0$. Since $\overline{R(K)} = Y$.

$\exists\, x \in X$, $\|Kx - Y\|^2 \le \frac{\varepsilon^2}{2}$.

$$\|Kx_\alpha - Y\|^2 \le J_\alpha(x_\alpha) \le J_\alpha(x) \le \varepsilon^2$$

for $\alpha \le \frac{\varepsilon^2}{2\|x\|^2}$

Thus $\|Kx_\alpha - Y\| \le \varepsilon$ for $\alpha < \frac{\varepsilon^2}{2\|x\|^2}$

So $\lim_{\alpha \to 0} Kx_\alpha = Y$

Morozov discrepancy principle

Asume $Kx = Y$ has a solution

$K : X \longrightarrow Y$ - linear, compact, one-to-one with dense range! Let $\alpha = \alpha(\delta)$ be such that

$x_{\alpha,\delta}$ solution of

$$\alpha x_{\alpha,\delta} + K^* K x_{\alpha,\delta} = K^* Y_\delta \quad (3)$$

Satisfies $\|K x_{\alpha,\delta} - y_\delta\| = \delta$. (4)

Note that (4) is <u>uniquely</u> solvable if

$\|y_\delta - y\| \leq \delta < \|y_\delta\|$ since.

$\lim_{\alpha\to\infty} \|K x_{\alpha,\delta} - y_\delta\| = \|y_\delta\| > \delta$.

$\lim_{\alpha\to 0} \|K x_{\alpha,\delta} - y_\delta\| = 0 < \delta$.

+ continuity and increasing character of

$\alpha \longrightarrow \|K x_{\alpha,\delta} - y_\delta\|$

<u>Theorem</u>. Assume all of the above, i.e., (3), (4). Then

I. $x_{\alpha,\delta} \longrightarrow x$ for $\delta \to 0$.

↳ The discrepancy principle is admissible.

II. if $x = K^* z$, $\|z\| \leq M$ then.

$\|x_{\alpha,\delta} - x\| \leq 2\sqrt{\delta}\cdot\sqrt{M}$.

When here we assumed $Kx = y$ has a solution!

Proof

Recall that $x_{\alpha,\delta}$ minimizes.

$$J(x) = \alpha(\delta)\|x\|^2 + \|Kx - Y_\delta\|^2$$

So, we have

$$\alpha(\delta)\|x_{\alpha,\delta}\|^2 + \delta^2 = J(x_{\alpha,\delta}) \leq J_\delta(x) = \alpha(\delta)\|x\|^2 + \|Y - Y_\delta\|^2$$

$$\leq \alpha(\delta)\|x\|^2 + \delta^2$$

Thus

$$\|x_{\alpha,\delta}\| \leq \|x\|$$

This implies.

$$\|x_{\alpha,\delta} - x\|^2 = \|x_{\alpha,\delta}\|^2 - 2(x_{\alpha,\delta}, x) + \|x\|^2 \leq$$

$$\leq 2(\|x\|^2 - (x_{\alpha,\delta}, x)) = 2(x - x_{\alpha,\delta}, x)$$

If $x = K^* z$ then.

$$\|x_{\alpha,\delta} - x\|^2 \leq 2(x - x_{\alpha,\delta}, K^* z) = 2(Y - K x_{\alpha,\delta}, z)$$

$$= 2(Y - Y_\delta, z) + 2(Y_\delta - K x_{\alpha,\delta}, z)$$

$$\leq 2\delta\|z\| + 2\delta\|z\| \leq 4\delta M.$$

↳ Thus II. follows.

Observe that since k is one to one

$N(k) = 0 \Rightarrow \overline{R(k^*)} = X.$

Thus there exists $\tilde{x} = k^* z$ with

$\|\tilde{x} - x\| \leq \varepsilon.$

$$\|x_{\alpha,\delta} - x\|^2 \leq 2(x - x_{\alpha,\delta}, x) =$$

$$= 2(x - x_{\alpha,\delta}, x - \tilde{x}) + 2(k^* z, x - x_{\alpha,\delta})$$

$$\leq 2\|x - x_{\alpha,\delta}\| \cdot \varepsilon + 4 \cdot \delta \|z\|.$$

$$\left(\|x_{\alpha,\delta} - x\| - \varepsilon\right)^2 \leq \varepsilon^2 + 4\delta\|z\|$$

Choose $\delta \leq \dfrac{\varepsilon^2}{4\|z\|}$

Computation of $\alpha(\delta)$ such that

$$\|k x_{\alpha,\delta} - y_\delta\| = \delta.$$ amounts to

applying Newton's method for finding roots of

$$\|k x_{\alpha,\delta} - y_\delta\|^2 - \delta^2 = 0.$$