# Radiation and initial-value problem

Radiation from $q(r,t) \in \mathbb{R}$ in homogeneous, isotropic, non-dispersive and non-attenuating media (e.g., free space)

The real-valued radiated fields satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(r,t) = q(r,t) \qquad (1)$$

where $c$ velocity of wave propagation in the host medium

$$\operatorname{supp} q = S_0 = \{ r \in \tau_0,\ t \in [0, T_0] \}, \quad \tau_0 \subset \mathbb{R}^3$$

$$\int_0^{T_0} dt \int_{\tau_0} d^3r \, |q(r,t)|^2 < +\infty.$$

- Results will be valid for dirac-delta sources and for $T_0 \longrightarrow \infty$.
- Any element in the kernel of $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ is also a solution of (1)

• Unique solution through initial conditions.

$$U(r,t)\Big|_{t=0} = U_0(r) \quad , \quad \frac{\partial}{\partial t} U(r,t)\Big|_{t=0} = U_1(r)$$

Cauchy conditions

$U_0, U_1$ arbitrary real valued functions.

• Causality

Solution $U_+$ to vanish prior to the turn-on time of the source $(t=0)$.

So the causal solution satisfies.

$$U_+(r,t)\Big|_{t=0} = 0 \quad , \quad \frac{\partial}{\partial t} U_+(r,t)\Big|_{t=0} = 0 . \qquad (2)$$

I. Problem (1) with conditions (2) is called the radiation problem

II Problem (1) with $9=0$ but arbitrary initial condition is called initial-value problem

## Fourier representation.

Assume $f(r,t)$ has a temporal Fourier transform

$$\mathcal{F}(r,\omega) = \int_{-\infty}^{\infty} f(r,t)\, e^{i\omega t}\, dt$$

and that $\mathcal{F}$ has a spatial Fourier transform

$$\tilde{\mathcal{F}}(\bar{s},\omega) = \int_{\mathbb{R}^3} \mathcal{F}(r,\omega)\, e^{-i\bar{s}\cdot\bar{r}}\, d^3 r$$

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Inversion:

$$f(\bar{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\bar{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$\mathcal{F}(\bar{r},\omega) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \tilde{\mathcal{F}}(\bar{s},\omega)\, e^{i\bar{s}\cdot\bar{r}}\, d^3 s.$$

Combination of temporal and spatial transforms

$$f(\bar{r},t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{\mathbb{R}^3} \tilde{\mathcal{F}}(\bar{s},\omega)\, e^{i(\bar{s}\cdot\bar{r}-\omega t)}$$

$$\tilde{\mathcal{F}}(\bar{s},\omega) = \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^3} f(\bar{r},t)\, e^{-i(\bar{s}\cdot\bar{r}-\omega t)}\, d^3 r$$

We have (in the sense of distributions)

$$\frac{\partial^n}{\partial t^n} f(\bar{r},t) \Longleftrightarrow (-i\omega)^n F(\bar{r},\omega)$$

$$\frac{\partial^n}{\partial x^n} F(\bar{r},\omega) \Longleftrightarrow (i s_x)^n \tilde{F}(\bar{s},\omega)$$

where $x$ is any of the Cartesian components of $r$

Dirac delta functions

$$\int_{-\infty}^{\infty} \delta(t)\,\phi(t)\,dt = \phi(0)$$

for smooth test functions $\phi(t), \chi(r)$

$$\int \delta(r)\,\chi(r)\,d^3r = \chi(0)$$

Fourier transforms of delta functions

$$\int_{-\infty}^{\infty} \delta(t)\,e^{i\omega t}\,dt = 1, \quad \int \delta(\bar{r})\,e^{-i\bar{s}\cdot\bar{r}}\,d^3r = 1.$$

Inverse transform (= any $E_i$)

$$\left.\begin{aligned} \delta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega \\ \delta(\bar{r}) &= \frac{1}{(2\pi)^3}\int e^{i\bar{s}\bar{r}}\,d^3s \end{aligned}\right\} \quad (*)$$

## 1D initial value problem

$$\begin{cases} \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right) U(z,t) = 0. \\ U(z,t)\Big|_{t=0} = U_0(z), \quad \dfrac{\partial}{\partial t} U(z,t)\Big|_{t=0} = U_1(z) \end{cases}$$

## General solution

$$U(z,t) = f(z-ct) + g(z+ct), \quad f,g \in C^2 \quad (3)$$

where $f, g$ need to be determined from the initial conditions by:

$$\begin{cases} f(z) + g(z) = U_0(z) \\ -\dfrac{\partial}{\partial z} f + \dfrac{\partial}{\partial z} g = \dfrac{1}{c} U_1(z) \end{cases} \Longleftrightarrow \begin{cases} \tilde{f}(s) + \tilde{g}(s) = \tilde{U}_0(k) \\ -is\tilde{f}(s) + is\tilde{g}(s) = \frac{1}{c}\tilde{U}_1(s) \end{cases}$$

where in this case

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(z) e^{-isz} dz \quad -$$ 1D Spatial Fourier transform

$$\left.\begin{aligned} \tilde{f}(s) &= \frac{1}{2}\left[\tilde{U}_0(s) + \frac{i}{cs}\tilde{U}_1(s)\right] \\ \tilde{g}(s) &= \frac{1}{2}\left[\tilde{U}_0(s) - \frac{i}{cs}\tilde{U}_1(s)\right] \end{aligned}\right\} \quad (4)$$

From (3),(4) we get the solution. is given by

$$U(z,t) = f(z-ct) + g(z+ct)$$

with.

$$f(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \tilde{U}_0(s) + \frac{i}{cs} \tilde{U}_1(s) \right) e^{isz} \, dz$$

$$g(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \tilde{U}_0(s) - \frac{i}{cs} \tilde{U}_1(s) \right) e^{isz} \, dz$$

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## Green functions.

A Green function of the wave equation is defined to be any solution of (1) for $I(\bar{r},t)$ given by $I(\bar{r},t) = \delta(\bar{r}-\bar{r}')\,\delta(t-t')$, where $r', t'$ are free parameters that can assume any value is space - time.

$g(r,r',t,t')$ - Green function if

$$\left( \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) g(\bar{r},\bar{r}',t,t') = \delta(\bar{r}-\bar{r}')\,\delta(t-t') \quad (5)$$

$g$ - Field radiated by an impulsive source located at $\bar{r}', t'$.

Homogeneity of space-time imply that

$$g(\bar{r}, \bar{r}', t, t') = g(\underbrace{\bar{r}-\bar{r}'}_{R}, \underbrace{t-t'}_{\tau})$$

(5) becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g(\bar{R}, \tau) = \delta(\bar{R})\,\delta(\tau) \qquad (6)$$

- Green function are not unique.. (Any function in the kernel of $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ is a solution)

- $g(\bar{R}, \tau)$ uniquely determined by Cauchy conditio

Fourier transform of (6)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g(\bar{R}, \tau) \Longleftrightarrow \left(-s^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\bar{s}, \omega)$$

where $s^2 = \bar{s}\cdot\bar{s}$.

Thus (6) implies

$$(-s^2 + k^2)\,\tilde{G}(\bar{s}, \omega) = 1 \quad , \quad k = \frac{\omega}{c} \text{ (wavenumber}$$

Thus we obtain

$$g(R,\tau) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int \frac{e^{i(\bar{\varsigma}\cdot\bar{R} - \omega\tau)}}{-\varsigma^2 + k^2} \, d^3\varsigma \qquad (7)$$

## Retarded and advanced Green functions

The Fourier integral representation (7) does not uniquely define a fundamental solution !!

Improper integral - Poles at $k = \pm \varsigma$.

- Necessary to deform the $\omega$-contour to avoid the poles ⟶ each contour will give a different Green function

## Causal Green function (Retarded Green function)

Vanishes for $\tau < 0$ ⟹ homogeneous Cauchy conditions at $\tau = 0$.

- Obtained from deforming the $\omega$ integration contour in (7) to lie above the poles.

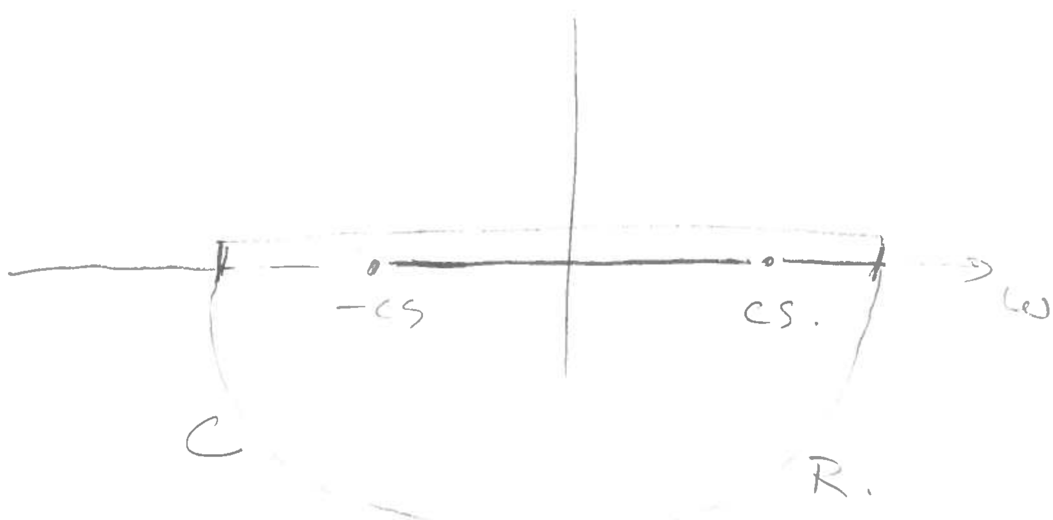
if $\tau < 0$, the contour can be closed in the upper half plane and from Cauchy's integral theorem we have that the integral vanishes and thus

$$g(\bar{R},\tau)=0 \quad \text{for } \tau<0 \quad (\text{Causality})$$

For $\tau>0$, we close the $\omega$ integration contour in the lower half plane and obtain.

$$g_+(\bar{R},\tau)=\frac{c^2}{(2\pi)^4}\int d^3s\, e^{i\bar{s}\cdot\bar{R}}\int_C \frac{e^{-i\omega\tau}}{\omega^2-c^2s^2}\,d\omega \qquad (8)$$

where $C$ is the causal contour.




Evaluate (8) by using residues theorem

$$g_+(\bar{R},\tau)=-\frac{c}{(2\pi)^3}\int e^{i\bar{s}\cdot\bar{R}}\,\frac{\sin(cs\tau)}{s}\,d^3s$$

Change of variable in the integral

Cartesian – spherical (Polar axis along $\bar{R}$)

$$d^3s = s^2 \sin\theta \, ds\, d\phi\, d\theta$$

$$\bar{s}\cdot\bar{R} = s \cdot R \cdot \cos\theta . \quad (\theta - \text{polar angle of } \bar{s})$$

$$g_+(\bar{R},\tau) = -\frac{c}{(2\pi)^2}\int_0^\infty s \sin(cs\tau)\int_0^\pi e^{isR\cos\theta}\sin\theta\, d\theta\, ds$$

$$= -\frac{c}{(2\pi)^2 R}\int_{-\infty}^{\infty}\sin(cs\tau)\sin(sR)\, ds.$$

$$= -\frac{1}{(2\pi)^2 R}\int_{-\infty}^{\infty}\sin(s\tau)\sin\left(s\frac{R}{c}\right) ds.$$

Euler identity

$$\sin(s\tau) = \frac{e^{is\tau} - e^{-is\tau}}{2i}$$

$$\sin\left(s\frac{R}{c}\right) = \frac{e^{isR/c} - e^{-isR/c}}{2i}$$

$$\sin(s\tau)\sin(sR/c) = \frac{e^{is(\tau-R/c)} + e^{-is(\tau-R/c)}}{4} - \frac{e^{is(\tau+R/c)} + e^{-is(\tau+R/c)}}{4}$$

From (*) we get, by using $\tau > 0$,

$$g_+(\bar{R},\tau) = -\frac{1}{4\pi}\frac{\delta(\tau - R/c)}{R} \qquad (9).$$

## Retarded Green function

$$(10)\quad \begin{cases} g_+(\bar{R},\tau) = -\dfrac{1}{4\pi}\dfrac{\delta(\tau - R/c)}{R}, & \tau > 0 \\ g_+(\bar{R},\tau) = 0 & \text{for } \tau < 0 \end{cases}$$

$g_+(\bar{R},\tau) = g_+(\bar{r}-\bar{r}', t-t')$ – field radiated from an impulsive source located at space-time point $\bar{r}', t'$. This field is observed at the space point $\bar{r}$ at $\tau = \frac{R}{c} \Rightarrow t = t' + R/c$

$t = t' + \frac{|\bar{r}-\bar{r}'|}{c}$, i.e. the observation is retarded by the distance between the two field points / divide by the velocity c of the host medium.

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◎ Another time-domain Green function

$$g_-(R,\tau) = -\frac{1}{4\pi}\frac{\delta(\tau + R/c)}{R}, \quad \tau < 0$$

$$g_-(R,\tau) = 0 \quad \text{for } \tau > 0$$

Important for inverse problems!

Green - function in the frequency domain

$$G(\bar{R},\omega) = \int_{-\infty}^{\infty} g(\bar{R},\tau)\, e^{i\omega\tau}\, d\tau$$

$$g(\bar{R},\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\bar{R},\omega)\, e^{-i\omega\tau}\, d\omega$$

$G(\bar{R},\omega)$ satisfies the Helmholtz equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] g(\bar{R},\tau) \longleftrightarrow \left[\nabla^2 + \frac{\omega^2}{c^2}\right] G(\bar{R},\omega)$$

Thus:

$$\left[\nabla^2 + k^2\right] G(\bar{R},\omega) = \delta(\bar{R}) \qquad (1\#)$$

$\downarrow$

Inhomogeneous Helmholtz equation!

Causality in the time domain implies the Somerfeld radiation condition (SRC) in the frequency domain.

$G_+(\bar{R},\omega)$ - Solution of (1) + (SRC)

$\hookrightarrow$ outgoing - wave.

$$\begin{cases} r\left(\frac{\partial U}{\partial r} - ikU\right) \longrightarrow 0 & 3D \\ \sqrt{r}\left(\frac{\partial U}{\partial r} - ikU\right) \longrightarrow 0 & 2D \end{cases}$$

By applying the Fourier transform we obtain

$$G_+(\bar{R}, \omega) = -\frac{1}{4\pi} \frac{e^{ikR}}{R}, \quad k = \frac{\omega}{c}$$

Observe that the retarded Green function $g_+$ satisfies

$$g_+(\bar{R}, \tau) = \frac{-1}{8\pi^2} \int_{-\infty}^{\infty} \frac{e^{-i\omega(\tau - R/c)}}{R} \, d\omega$$

↳ Superposition of spherical time-harmonic waves $-\frac{1}{4\pi} \frac{e^{ik(R - c\tau)}}{R}$ which propagate outward from $\bar{R} = 0$ with increasing time

Similarly

$$G_-(\bar{R}, \omega) = -\frac{1}{4\pi} \frac{e^{-ikR}}{R} = G_+^*(\bar{R}, \omega), \quad k \in \mathbb{R}$$

↳ incoming-wave Green function

# 1D Example

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] g(z,t) = \delta(z)\,\delta(t) \qquad (12)$$

Fourier time transform applied to (12)

$$\left[\frac{\partial^2}{\partial z^2} + k^2\right] G(z,\omega) = \delta(z) \qquad (13)$$

Fourier spatial transform on (13)

$$\tilde{G}(s,\omega) = \frac{1}{-s^2+k^2}$$

Thus.

$$G(z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{isz}}{-s^2+k^2}\,ds \qquad (14)$$

$\hookrightarrow$ Improper integral with poles $s = \pm k$

Further

$$g_+(z,\tau) = \frac{1}{2\pi}\int_C G_+(z,\omega)\, e^{-i\omega\tau}\,d\omega \qquad (15)$$

where $C$ is in the upper half plane ($\mathrm{Im}\,\omega > 0$) for a causal Green function ($g_+(z,\tau) = 0$, $\tau < 0$)

Poles of (14) are at $s = \pm k$. One lies in the upper half plane and lower half plane respectively.

The integrand goes to zero exponentially fast in the upper half plane if $z > 0$ and in the lower half plane if $z < 0$ the integral gives by using residues

$$G_+(z,\omega) = \frac{1}{2\pi} 2\pi i \cdot \frac{e^{ikz}}{-2k}$$

$$= -\frac{i}{2k} e^{ikz}, \quad z > 0$$

$$G_+(z,\omega) = \frac{-1}{2\pi} 2\pi i \frac{e^{-ikz}}{\cdot 2k}, \quad z < 0$$

$$= -\frac{i}{2k} e^{-ikz} \quad (\text{Index} = -1 \text{ in this case})$$

$$G_+(0,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k^2 - s^2} ds = \frac{1}{2\pi} 2\pi i \frac{-1}{2k} = \frac{-i}{2k}.$$

Check

$$\frac{\partial}{\partial z} G_+(z,\omega) = \frac{1}{2} e^{ik|z|} \operatorname{Sgn}(z), \quad \frac{\partial^2}{\partial z^2} G_+(z,\omega) = \frac{ik}{2} e^{ik|z|} \operatorname{Sgn}^2(z) + \frac{1}{2} e^{ik|z|} \operatorname{Sgn}' z$$

$\mathrm{Sgn}\, z = 2 \cdot H(z) - 1$. implies the test

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Retarded causal time-domain Green function.

$$g_+(z,\tau) = \frac{1}{2\pi} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} -\frac{i}{2k} e^{ik|z|} e^{-i\omega\tau}\, d\omega =$$

$$= -\frac{ic}{4\pi} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{e^{i\frac{\omega}{c}|z|}}{\omega} \cdot e^{-i\omega\tau}\, d\omega$$

Pole at $\omega = 0$. Exponentially fast decay in upper half plane if $|z| > c\tau$ and in the lower half plane if $|z| < c\tau$, Residues imply

$$g_+(z,\tau) = -\frac{c}{2} H(c\tau - |z|)$$

where $H$ is the Heaviside function

Green-function solution to the radiation problem.

$$\left[\nabla_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] U_+(\bar{r}', t') = Q(\bar{r}', t') \qquad (1)$$

$$\left[\nabla_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] g(\bar{r}-\bar{r}', t-t') = \delta(\bar{r}-\bar{r}')\,\delta(t-t') \qquad (2)$$

In fact I use. $Q(\bar{r}, t) = \delta(\bar{r}-\bar{r}')\,\delta(t-t') \cdot Q(\bar{r}', t')$.

where $g$ is an unspecified green function of the wave equation.

$(\bar{r}, t)$ — fixed parameter in $R = \{\bar{r} \in \tau,\ t \in (t_0, t_1)\}$

Look for a solution of the radiation problem in $R$.

Multiply (1) by $g$ and (2) by $U_+$ and substract.

to obtain

$$U_+(\bar{r}', t')\,\nabla^2_{r'} g(\bar{r}-\bar{r}', t-t') - g(\bar{r}-\bar{r}', t-t')\,\nabla^2_{r'} U_+(\bar{r}', t')$$

$$-\frac{1}{c^2}\left\{U_+(\bar{r}', t')\frac{\partial^2}{\partial t'^2} g(\bar{r}-\bar{r}', t-t') - g(\bar{r}-\bar{r}', t-t')\frac{\partial^2}{\partial t'^2} U_+(\bar{r}', t')\right\}$$

$$= U_+(\bar{r}', t')\,\delta(\bar{r}-\bar{r}')\,\delta(t-t') - g(\bar{r}-\bar{r}', t-t')\,Q(\bar{r}', t') \qquad (3)$$

Let the first, second and third terms in the above relation be denoted by

$$I_1(\bar{r},\bar{r}',t,t'),\quad I_2(\bar{r},\bar{r}',t,t'),\quad I_3(\bar{r},\bar{r}',t,t')$$

Then (3) becomes, after integration over $R$

$R = \{ r' \in \tau,\ t' \in (t_0, t_1) \}$

$$\underbrace{\int_{t_0}^{t_1} dt' \int_\tau d^3r'\, I_1}_{x_1(\bar{r},t)} + \underbrace{\int_{t_0}^{t_1} dt' \int_\tau d^3r'\, I_2}_{x_2(\bar{r},t)} = \underbrace{\int_{t_0}^{t_1} dt' \int_\tau d^3r'\, I_3}_{x_3(\bar{r},t)} \qquad (4)$$

$$x_1(\bar{r},t) = \int_{t_0}^{t_1} dt' \int_\tau d^3r'\ \nabla\cdot[U_+ \nabla g - g \nabla U_+] =$$

$$= \int_{t_0}^{t_1} \int_{\partial\tau} \left( U_+ \frac{\partial}{\partial n'} g - g \frac{\partial}{\partial n'} U_+ \right) ds'\, dt' \qquad (5)$$

where we have used $\nabla\cdot(\phi \nabla V) = \nabla\phi\cdot\nabla V + \phi \nabla^2 V$

$$x_2(\bar{r},t) = -\frac{1}{c^2} \int_{t_0}^{t_1} \int_\tau \frac{\partial}{\partial t'} \left( U_+ \frac{\partial}{\partial t'} g - g \frac{\partial}{\partial t'} U_+ \right) d^3r'\, dt' =$$

$$= -\frac{1}{c^2} \int_\tau \left( U_+ \frac{\partial}{\partial t'} g - g \frac{\partial}{\partial t'} U_+ \right) \Bigg|_{t'=t_0}^{t_1} d^3r' \qquad (6)$$

$$x_3(\bar{r},t) = U_+(r,t) - \int_{t_0}^{t_1}\int_{\tau} g(\bar{r}-\bar{r}', t-t')\, \varrho(r',t')\, d^3r'\,dt'$$

$$\text{if } (\bar{r},t) \in R. \qquad (7)$$

$$x_3(\bar{r},t) = -\int_{t_0}^{t_1}\int_{\tau} g(\bar{r}-\bar{r}', t-t')\, \varrho(\bar{r}',t')\, d^3r'\,dt'$$

otherwise.

- Solution $U_+$ valid for all space-time.

Select $g = g_+$ in (4) above and study $x_1, x_2, x_3$ when $\tau \to \infty$, $t_0 \to -\infty$, $t_1 \to \infty$.

$x_2(\bar{r},t) \to 0$ because causality of $U_+$ and $g_+$

$\lim_{t_1\to\infty} \lim_{\tau\to\infty} x_1(\bar{r},t) = 0$ as well from causality

From (4) we have that $\lim_{t_1\to\infty} \lim_{\tau\to\infty} x_3(\bar{r},t) = 0$

Thus, from (7) we get.

$$U_+(\bar{r},t) = \int_0^{T_0}\int_{\tau_0} g_+(\bar{r}-\bar{r}', t-t')\, \varrho(\bar{r}',t')\, d^3r'\,dt'$$

or

$$U_+(\bar{r},t) = -\frac{1}{4\pi}\int_{\tau_0} \frac{\varrho\left(\bar{r}', t - \frac{|r-r'|}{c}\right)}{|r-\bar{r}'|}\, d^3r'$$

for $\bar{r} \in \mathbb{R}^3$, $t \in \mathbb{R}$

$U_+(\bar{r}, t)$ is a superposition of expanding spherical waves $g_+(\bar{r}-\bar{r}', t-t')$ each weighted by the amplitude of the source $\varrho(\bar{r}', t')$.

Thus the contribution from any point source is a delta function over the surface.

$$|\bar{r}-\bar{r}'| = c(t-t')$$

$$\Downarrow$$

<u>Light cone</u>

and thus expands outward with velocity $c$ from each source point $\bar{r}'$

If $\tau_0$ is a sphere of radius $a$ centered in O then at any given time $t$ the field is identically zero outside a sphere $V_+(t)$ centered at the O and radius $R_+(t) = a + ct$.

If $\varrho = 0$ for $t \geq T_0$ the field vanishes for all time, $t > T_0 + a/c$ inside a sphere $V_-(t)$ centered at O and with Radius $R_-(t) = c(t-T_0) - a$

Ball of energy in the outward direction!!

Exercise $\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] U_+(z,t) = g(z,t)$

Supp $g = [-a_0, a_0]$, $t \in [0, T_0]$ and we require the field to be causal.

Show that

$$U_+(z,t) = -\frac{c}{2}\int_{-a_0}^{a_0}\int_0^{t-\frac{|z-z'|}{c}} g(z',t')\,dt'\,dz'$$