ON THE NUMERICAL SOLUTION OF A NONLINEAR, NON – SMOOTH EIGENVALUE PROBLEM OR WHEN BINGHAM MEETS BRATU:

AN OPERATOR – SPLITTING APPROACH

Abstract

Some years ago, we suggested to a colleague looking for nonlinear saddle-point problems with multiple solutions (in order to test mountain-pass based solution methods) to have a look at the following elliptic one:

(BBPV) $$\begin{cases} \text{Find } \{u,\lambda\} \in H_0^1(\Omega)\times \mathbf{R}_+ \text{ such that} \\ \mu\int_\Omega \nabla u \,.\, \nabla(v-u)dx + \tau_y[\int_\Omega |\nabla v|dx - \int_\Omega |\nabla u|dx] \geq \lambda\int_\Omega e^u\ (v-u)dx, \forall v \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^2$, μ and τ_y being both > 0.

(BBPV) is nothing, but the variational formulation of the following nonlinear, non-smooth Dirichlet problem

(BBPE) $$\begin{cases} -\mu\nabla^2 u + \tau_y \partial j(u) \ni \lambda e^u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\partial j(u)$ denotes the sub-differential at u of the convex functional j: $H_0^1(\Omega) \rightarrow \mathbf{R}$ defined by

$j(v) = \int_\Omega |\nabla v|dx.$ Suppose that $\tau_y = 0$ in the above formulations, then the above problem reduces to the celebrated ***Bratu-Gelfand*** problem

$$\begin{cases} -\mu\nabla^2 u = \lambda e^u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

On the other hand, if, in (BBPV) and (BBPE), one replaces λe^u by a constant ϖ, the resulting inequalities and equations model the flow of a ***Bingham visco-plastic medium*** of viscosity μ and plasticity yield τ_y in an infinitely long cylinder of cross-section Ω, with ϖ, and u denoting the (algebraic) pressure drop per unit length and the flow axial velocity, respectively.

Problem (BBPV), (BBPE) has clearly the flavor of a non-smooth nonlinear eigenvalue problem for an elliptic operator. The numerical solution of such problems by minimax (mountain-pass) methods has been investigated by our colleagues ***Xudong Yao*** and ***Jianxin Zhou***. Our goal in this lecture is to present a conceptually simpler methodology based on ***operator-splitting***: The resulting algorithms are natural generalizations of the ***inverse power method*** for symmetric matrix eigenvalue computation.

The results of numerical experiments performed by our collaborator ***F. Foss*** will be presented.