The Cauchy-Euler Equation

The differential equation

$$x^2y'' + axy' + by = 0 \text{ for } x > 0 \tag{1}$$

or its equivalent

$$y'' + a\frac{1}{x}y' + b\frac{1}{x^2}y = 0 \text{ for } x > 0 \tag{1}$$

where each of a and b is a number is known as the Cauchy-Euler differential equation. Its assiciated polynomial is the function Q given by

$$Q(r) = r^2 + (a-1)r + b$$

for all complex numbers r.

Theorem. (1) If Q has zeros r_1 and r_2 ($r_1 \neq r_2$), then y is a solution to (1) if and only if

$$y = c_1x^{r_1} + c_2x^{r_2}$$

for all $x > 0$ and some pair of numbers (c_1, c_2).

(2) If Q has only one zero r, then y is a solution to (1) if and only if

$$y = c_1x^r + c_2x^r \ln x$$

for all $x > 0$ and some pair of numbers (c_1, c_2).

(3) If Q has complex zeros $\alpha + \beta i$ and $\alpha - \beta i$ where each of α and β is real and $\beta \neq 0$,then y is a solution to (1) if and only if

$$y = c_1x^\alpha \cos(\beta \ln x) + c_2x^\alpha \sin(\beta \ln x)$$

for all $x > 0$ and some pair of numbers (c_1, c_2).

Suggestion for Proof. Let u and y be related by

$$u(x) = y(e^x) \text{ for all } x$$

or

$$y(x) = u(\ln x) \text{ for } \dot{x} > 0$$

and let

$$t = \ln x \text{ or } x = e^t.$$

Show that y satisfies (1) if and only if u satisfies the constant coefficient equation

$$u''(t) + (a-1)u'(t) + bu(t) = 0 \text{ for all } t. \tag{2}$$