EMCF Quiz 9 Problems 6 and 7

The given system is

$$\begin{aligned} x - 2y &= 1 \\ x - y + kz &= -2 \\ ky + 4z &= 6. \end{aligned}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 1 & -1 & k & -2 \\ 0 & k & 4 & 6 \end{pmatrix}.$$

Row reducing, we have

$$\xrightarrow[-R_1 + R_2 \to R_2]{} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & k & -3 \\ 0 & k & 4 & 6 \end{pmatrix} \xrightarrow[-kR_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & k & -3 \\ 0 & 0 & 4 - k^2 & 3k + 6 \end{pmatrix}.$$

If $k^2 \neq 0$ (i.e. if $k \neq 2$ and $k \neq -2$), backward substitution substitution shows that there is a unique solution,

$$(x, y, z) = \left(\frac{k + 10}{k - 2}, \frac{6}{k - 2}, \frac{3}{2 - k} \right).$$

If $k = -2$, the last matrix becomes

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and backward substitution shows that (x, y, z) is a solution if and only if

$$(x, y, z) = (4a - 5, 2a - 3, a)$$

for some number a.There are infinitely many solutions.

If $k = 2$, the matrix

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & k & -3 \\ 0 & 0 & 4 - k^2 & 3k + 6 \end{pmatrix}$$

becomes

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 12 \end{pmatrix}.$$

The last of the three implied equations is

$$0 = 12$$

so the given system has no solution.