# Section 1.1
## Basic Terminology

### Basic Terminology.

A **differential equation** (DE) is an equation that contains an unknown function and at least one of its derivatives.

For now, the unknown function will usually be denoted by $y$.

The **order** of a differential equation is the order of the highest derivative of the unknown function.

An **ordinary** differential equation (ODE) contains only ordinary derivatives of the unknown function. The domain of the unknown function is a set of real numbers.

A **partial** differential equation (PDE) contains at least one partial derivative of the unknown function. The domain of the unknown function is a subset of $\mathbb{R}^n$ for some $n > 1$.

**Examples**: When $y$ is the unknown function,

$$y' = \frac{x^2 y - y}{y+1} \text{ is a first order ODE.}$$

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3 \text{ is a second order ODE.}$$

When $u$ is the unknown function,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ is a second order PDE.}$$

When $y$ is the unknown function,

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 3e^{-x} \text{ is a third order ODE.}$$

When $y$ is the unknown function,

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{d^3}{dx^3}[e^{-2x}] \text{ is a second order ODE.}$$

Even though you see a third order derivative, it is not of the unknown function.

While there are a few partial differential equations in this first section, this is a course in **ordinary** differential equations with some linear algebra.

An ordinary differential equation is one that fits the form

$$F(x,y,y',\ldots,y^{(n)}) = G(x,y,y',\ldots,y^{(m)}).$$

Some special cases are

$$F(x,y,y',\ldots,y^{(n)}) = G(x),$$

$$F(x,y,y',\ldots,y^{(n)}) = 0,$$

and

$$y' = F(x,y).$$

## Intervals

**Definition**. An **interval** is a connected set of real numbers containing more than one number.

## Definition of a Solution

Saying that a function $u$ is a **solution** to the ordinary differential equation

$$F(x,y,y',\ldots,y^{(n)}) = G(x,y,y',\ldots,y^{(m)})$$

means that $u$ is a function whose domain is an interval and

$$F(x,u(x),u'(x),\ldots,u^{(n)}(x)) \ = G(x,u(x),u'(x),\ldots,u^{(m)}(x))$$

for all numbers $x$ in that interval.

**Note**: At this point, we will not be too concerned with domains. In the problems, to check whether or not a given function $u$ is a solution to a differential equation of the form

$$F(x,y,y',\ldots,y^{(n)}) = G(x,y,y',\ldots,y^{(m)}),$$

compute and simplify each of $F(x,u(x),u'(x),\ldots,u^{(n)}(x))$ and $G(x,u(x),u'(x),\ldots,u^{(m)}(x))$. If the simplified expressions are the same, then $u$ is a solution. Otherwise, it is not a solution.

**Example Problem**: Suppose that

$$u(x) = \sin 2x$$

Show that $u$ is a solution to

$$y'' + 4y = 0.$$

**Solution**: Replace $y$ with $u(x)$ and $y''$with $u''(x)$ and show that the result reduces to zero.

$$u(x) = \sin 2x$$

then

$$u'(x) = 2\cos 2x$$

and

$$u''(x) = -4\sin 2x$$

so

$$u''(x) + 4u(x) = -4\sin 2x + 4\sin 2x = 0.$$

**Example Problem**: Suppose that

$$u(x) = \sin 3x.$$

Show that $y$ is not a solution to

$$y'' + 4y = 0.$$

**Solution**: Replace $y$ with $u(x)$ and $y''$with $u''(x)$ and show that the result reduces to something other than zero.

$$u(x) = \sin 3x$$

then

$$u'(x) = 3\cos 3x$$

and

$$u''(x) = -9\sin 3x$$

so

$$u''(x) + 4u(x) = -9\sin 3x + 4\sin 3x = -5\sin 3x.$$

$-5\sin 3x$ is not the zero function so this function ( $u$ where $u(x) = \sin 3x$ ) is not a solution to the differential equation.

## Notation

- The symbol $y$ will often be used to denote a function.
- The symbol $x$ will often be used to denote a number.
- The symbol $x$ will also often be used to denote the identity function.

$$x(t) = t$$

.

- For example, saying $y$ is the function such that

$$y(x) = x^2 \text{ for all numbers } x$$

is equivalent to saying that

$$y = x^2.$$

**Example Problem**:Consider the differential equation

$$y'' - 2y' - 8y = 4e^{2x}.$$

and the functions $u$ and $v$ given by

$$u = 2e^{4x} - \frac{1}{2}e^{2x}$$

and

$$v = 2e^{2x} - \frac{1}{2}e^{4x}.$$

Is $u$ a solution to the DE? Is $v$ a solution to the DE?

**Solution**.

$$u = 2e^{4x} - \frac{1}{2}e^{2x}$$

$$u' = 8e^{4x} - e^{2x}$$

$$u'' = 32e^{4x} - 2e^{2x}$$

$$\begin{aligned} u'' - 2u' - 8u &= (32e^{4x} - 2e^{2x}) - 2(8e^{4x} - e^{2x}) - 8(2e^{4x} - \frac{1}{2}e^{2x}) \\ &= (32 - 16 - 16)e^{4x} + (-2 + 2 + 4)e^{2x} \\ &= 4e^{2x} \end{aligned}$$

This is the right side of the given differential equation, so $u$ is a solution.

Remember the DE is

$$y'' - 2y' - 8y = 4e^{2x}.$$

$$v = 2e^{2x} - \frac{1}{2}e^{4x}$$

$$v' = 4e^{2x} - 2e^{4x}$$

$$v'' = 8e^{2x} - 8e^{4x}$$

$$\begin{aligned} v'' - 2v' - 8v &= (8e^{2x} - 8e^{4x}) - 2(4e^{2x} - 2e^{4x}) - 8(2e^{2x} - \frac{1}{2}e^{4x}) \\ &= (-8 + 4 + 4)e^{4x} + (8 - 8 - 16)e^{2x} \\ &= -16e^{2x} \end{aligned}$$

This is not the right side of the differential equation which is $4e^{2x}$, so $v$ is not a solution.

**Example Problem**: Consider the differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y}$$

where $u$ is the unknown function. Let the functions $z$ and $w$ given by

$$z = \sin(2x + y)$$

and

$$w = e^{2x+y}.$$

Is $z$ a solution to the DE? Is $w$ a solution to the DE?

**Solution**.

$$z = \sin(2x + y)$$

$$\frac{\partial z}{\partial x} = \cos(2x + y)(2) = 2\cos(2x + y)$$

and

$$\frac{\partial z}{\partial y} = \cos(2x + y)(1) = \cos(2x + y)$$

so

$$2\frac{\partial z}{\partial y} = 2\cos(2x + y).$$

The DE is

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y}.$$

Since

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y},$$

it follows that $z$ is a solution.

$$w = e^{2x+y}$$

$$\frac{\partial w}{\partial x} = e^{2x+y}(2) = 2e^{2x+y}$$

and

$$\frac{\partial w}{\partial y} = e^{2x+y}(1) = e^{2x+y}$$

so

$$2\frac{\partial w}{\partial y} = 2e^{2x+y}$$

The DE is

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y}.$$

Since

$$\frac{\partial w}{\partial x} = 2\frac{\partial w}{\partial y},$$

it follows that $w$ is a solution.

## Note

Some differential equations can be solved just by integrating.

The solutions to

$$y' = f(x)$$

are given by

$$y(x) = \int f(x)dx + C$$

The solutions to

$$y'' = f(x)$$

are given by

$$y(x) = \int\int f(x)dxdx + C_1x + C_2.$$

## Example

Find all solutions to

$$y' = 2x + e^{2x}.$$

**Solution**:

$$\int(2x + e^{2x})dx = x^2 + \frac{1}{2}e^{2x}$$

so

$$y = x^2 + \frac{1}{2}e^{2x} + C \text{ for some number } C.$$

If

$$y = x^2 + \frac{1}{2}e^{2x} + C \text{ for some number } C$$

then, differentiating,

$$y' = 2x + e^{2x}.$$

## Example

Find all solutions to

$$y'' = \sin 2x$$

**Solution**: Suppose that $y$ is a solution. Then integrating

$$y' = -\frac{1}{2}\cos 2x + C_1$$

and integrating again

$$y = -\frac{1}{4}\sin 2x + C_1 x + C_2 \qquad \#$$

for some pair of numbers $C_1$ and $C_2$. If $y$ is given by (#) then differentiating twice produces

$$y'' = \sin 2x$$

## Example

Some differential equations have solutions that are exponential functions. Consider

$$y'' - y' - 6y = 0$$

and suppose that

$$y = e^{rx}.$$

Then

$$y' = re^{rx} \text{ and } y'' = r^2 e^{rx}$$

so

$$y'' - y' - 6y = r^2 e^{rx} - re^{rx} - 6e^{rx} = (r^2 - r - 6)e^{rx}.$$

This is $0$ if and only if $r^2 - r - 6 = 0$ or $(r+2)(r-3) = 0$ if and only if $r = -2$ or $r = 3$. Thus $y$ is a solution if and only if

$$r = -2 \text{ or } r = 3.$$

## Example

Some differential equations have solutions that are power functions. Consider the differential equation

$$x^2 y'' - 5xy' + 8y = 0$$

and suppose that

$$y = x^r.$$

Then

$$y' = rx^{r-1} \text{ and } y'' = r(r-1)x^{r-2}$$

so

$$\begin{aligned} x^2 y'' - 5xy' + 8y &= x^2 r(r-1)x^{r-2} - 5xrx^{r-1} + 8x^r \\ &= r^2 x^r - rx^r - 5rx^r + 8x^r \\ &= (r^2 - 6r + 8)x^r. \end{aligned}$$

This is $0$ on an interval if and only if $r^2 - 6r + 8 = 0$ or $(r-2)(r-4) = 0$ if and only if $r = 2$ or $r = 4$. Thus $y$ is a solution if and only if

$$r = 2 \text{ or } r = 4.$$

**Additional Examples**: See Section 1.1 of the text and the notes presented on the board in class.

**Suggested Problems**: Do the odd numbered problems for Section 1.1 The answers are posted on Dr. Walker's web site.