Engineering Mathematics
Section 1.1

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Basic Terminology

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A **differential equation** (DE) is an equation [ ]

For now, the unknown function will usually be denoted by [ ]

The **order** of a differential equation is [ ]

An **ordinary** differential equation (ODE) contains only [ ] The domain of the unknown function is a set of [ ]

A **partial** differential equation (PDE) contains at least [ ]. The domain of the unknown function is a subset of [ ]

Examples: When y is the unknown function,

$$y' = \frac{x^2 y - y}{y+1} \text{ is a [ ] .}$$

$$x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3 \text{ is a [ ] .}$$

When u is the unknown function,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ is a [ ] .}$$

When y is the unknown function,

$$\frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} = 3e^{-x} \text{ is a [ ] .}$$

When y is the unknown function,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{d^3}{dx^3}[e^{-2x}] \text{ is a [ ] .}$$

Even though you see a third order derivative, it is not of the [ ].

While there are a few partial differential equations in this first section, this is a course in **ordinary** differential equations with some linear algebra.

An ordinary differential equation is one that fits the form

$$F(x, y, y', \ldots, y^{(n)}) = G(x, y, y', \ldots, y^{(m)}).$$

Some special cases are

$$F(x, y, y', \ldots, y^{(n)}) = G(x),$$

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

and

$$y' = F(x, y).$$

Intervals

Definition. An **interval** is a .

Definition of a Solution

Saying that a function u is a **solution** to the ordinary differential equation

$$F(x, y, y', \ldots, y^{(n)}) = G(x, y, y', \ldots, y^{(m)})$$

means that u is a function whose domain is an interval and

$$F(x, u(x), u'(x), \ldots, u^{(n)}(x)) = G(x, u(x), u'(x), \ldots, u^{(m)}(x))$$

for all numbers x in that interval.

Note: At this point, we will not be too concerned with domains. In the problems, to check whether or not a given function u is a solution to a differential equation of the form

$$F(x, y, y', \dots, y^{(n)}) = G(x, y, y', \dots, y^{(m)}),$$

compute and simplify each of $F(x, u(x), u'(x), \dots, u^{(n)}(x))$ and $G(x, u(x), u'(x), \dots, u^{(m)}(x))$. If the simplified expressions are the same, then u is a solution. Otherwise, it is not a solution.

Example Problem: Suppose that

$$u(x) = \sin 2x$$

Show that u is a solution to

$$y'' + 4y = 0.$$

Solution: Replace y with $u(x)$ and $y''$with $u''(x)$ and show that the result reduces to zero.

$$u(x) = \sin 2x$$

then

$$u'(x) =$$

and

$$u''(x) =$$

so

$$u''(x) + 4u(x) = () + 4() = . =$$

Example Problem: Suppose that

$$u(x) = \sin 3x.$$

Show that y is not a solution to

$$y'' + 4y = 0.$$

Solution: Replace y with $u(x)$ and $y''$with $u''(x)$ and show that the result reduces to something other than zero.

$$u(x) = \sin 3x$$

then

$$u'(x) = ()$$

and

$$u''(x) = ()$$

so

$$u''(x) + 4u(x) = () + 4() == .$$

$()$ is not the zero function so this function u where $u(x) = \sin 3x$ is not a solution to the differential equation.

Notation

- The symbol y will often be used to denote a function.

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- For example, saying y is the function such that

$$y(x) = x^2 \text{ for all numbers } x$$

is equivalent to saying that

$$y = x^2.$$

Example Problem: Consider the differential equation

$$y'' - 2y' - 8y = 4e^{2x}.$$

and the functions u and v given by

$$u = 2e^{4x} - \frac{1}{2}e^{2x}$$

and

$$v = 2e^{2x} - \frac{1}{2}e^{4x}.$$

Is u a solution to the DE? Is v a solution to the DE?

Solution.

$$u = 2e^{4x} - \frac{1}{2}e^{2x}$$

$$u' =$$

$$u'' =$$

$$\begin{aligned} u'' - 2u' - 8u &= () - 2() - 8() \\ &= ()e^{4x} + ()e^{2x} \\ &= \end{aligned}$$

This is is not the right side of the given differential equation, so u is is not a solution.

Remember the DE is

$$y'' - 2y' - 8y = 4e^{2x}.$$

$$v = 2e^{2x} - \frac{1}{2}e^{4x}$$
$$v' =$$
$$v'' =$$

$$\begin{aligned} v'' - 2v' - 8v &= () - 2() - 8() \\ &= ()e^{4x} + ()e^{2x} \\ &= \end{aligned}$$

This is is not the right side of the differential equation, so v is is not a solution.

Example Problem: Consider the differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y}$$

where u is the unknown function. Let the functions z and w given by

$$z = \sin(2x + y)$$

and

$$w = e^{2x+y}.$$

Is z a solution to the DE? Is w a solution to the DE?

Solution.

$$z = \sin(2x + y)$$

$$\frac{\partial z}{\partial x} =$$

and

$$\frac{\partial z}{\partial y} =$$

so

$$2\frac{\partial z}{\partial y} = .$$

Since

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y},$$

it follows that z is a solution.

$$w = e^{2x+y}$$

$$\frac{\partial w}{\partial x} =$$

and

$$\frac{\partial w}{\partial y} =$$

so

$$2\frac{\partial w}{\partial y} = 2e^{2x+y}$$

Since

$$\frac{\partial w}{\partial x} = 2\frac{\partial w}{\partial y},$$

it follows that w is a solution.

Note

Some differential equations can be solved just by integrating.
The solutions to

$$y' = f(x)$$

are given by

$$y(x) = \int f(x)dx + C$$

The solutions to

$$y'' = f(x)$$

are given by

$$y(x) = \int \int f(x)dxdx + C_1 x + C_2.$$

Example

Find all solutions to

$$y' = 2x + e^{2x}.$$

Solution:

$$\int (2x + e^{2x})dx =$$

so

$$y = .$$

Conversely, if

$$y =$$

then, differentiating,

$$y' = 2x + e^{2x}.$$

Example

Find all solutions to

$$y'' = \sin 2x$$

Solution: Suppose that y is a solution. Then integrating,

$$y' =$$

and integrating again,

$$y = \qquad (\#)$$

for some pair of numbers C_1 and C_2. If y is given by $(\#)$ then differentiating twice produces

$$y'' = \sin 2x$$

Example

Some differential equations have solutions that are exponential functions. Consider

$$y'' - y' - 6y = 0$$

and suppose that

$$y = e^{rx}.$$

Then

$$y' = \quad \text{and } y'' =$$

so

$$y'' - y' - 6y = () - () - 6() = .$$

This is 0 if and only if $() = 0$ or$()() = 0$ if and only if $r =$ or $r =$. Thus y is a solution if and only if

$$r = \quad \text{or } r = .$$

Example

Some differential equations have solutions that are power functions. Consider the differential equation

$$x^2y'' - 5xy' + 8y = 0$$

and suppose that

$$y = x^r.$$

Then

$$y' = \text{ and } y'' =$$

so

$$\begin{aligned} x^2y'' - 5xy' + 8y &= x^2() - 5x() + 8() \\ &= ()x^r - ()x^r - ()x^r + ()x^r \\ &= ()x^r. \end{aligned}$$

This is 0 on an interval if and only if $() = 0$ or $()() = 0$ if and only if $r =$ or $r =$. Thus y is a solution if and only if

$$r = \text{ or } r = .$$

Additional Examples: See Section 1.1 of the text.

Suggested Problems: Do the odd numbered problems for Section 1.1
The answers are posted on Dr. Walker's web site.