Section 1.2

Parameterized Families

We will introduce the notion of a parameterized family of function by way of some examples.

Example 1. Suppose that $\mathcal{F} = \{f | f(x) = 3x + C$ for all x and some number $C\}$. This is an example of a one-parameter family of functions. It consists of all straight line functions with slope 3. The parameter C determines the y-intercept.

Example 2. Suppose that

$$\mathcal{F} = \{f | f(x) = C_1x^2 + C_2 \text{ for all } x \text{ and some pair of number } C_1 \text{ and } C_2\}.$$

This set of quadratic functions is an example of a two-parameter family of functions. The parameters C_1 and C_2 determine the spread and vertex of the parabola and whether it opens upward or downward.

General Solutions, **Singular Solutions**, **and Particular Solutions**

The set of solutions to an n-th order differential equation is often contained in an n-parameter family of functions. An expression that indicates this family is sometimes called the **general solution** of the differential equation.

Example 3. We will learn later in this course that y is a solution to

$$y'' + 4y = 0 \qquad 1$$

on the set of real numbers if and only if

$$y = C_1 \cos 2x + C_2 \sin 2x \qquad 2$$

for some pair of numbers C_1 and C_2 and all real numbers x. In this case (2) is called the general solution to (1).

In some cases, most solutions to a differential equation will be containedn in members of a parameterized family. Those solutions, if any, that are not members of the family are known as **singular solutions**.

Example 4. Most solutions of

$$y' = y^2$$

are contained in members of the one-parameter family indicated by

$$y = \frac{1}{C - x}.$$

However, the function given by

$$y = 0$$

is also a solution. It is an example of a singular solution.

If specific values are assigned to the parameters in a parameterized family of solutions, the result is known as a **particular solution**.

Example 5. In view of Example 3, the function y given by

$$y = 3\cos 2x - 5\sin 2x$$

is a particular solution to

$$y'' + 4y = 0.$$

Finding the Differential Equation

Sometimes a differential equation can be reconstructed from a parameterized family of solutions. If the family contains n parameters, we look for an n-th order differential equation. Given the n-parameter family, we compute derivatives through order n and use these equations to eliminate the parameters.

Example 6. Find a differential equation for the family indicated by

$$y = Cx^3 + 1$$

Solution. From

$$y = Cx^3 + 1 \qquad 3$$

we have

$$y' = 3Cx^2. \qquad 4$$

From (4) we have

$$C = \frac{y'}{3x^2}.$$

Using this expression for C, equation (3) becomes

$$y = \frac{y'}{3x^2}x^3 + 1 \text{ so } y = \frac{y'}{3}x + 1.$$

Thus

$$3y = xy' + 3 \text{ so } xy' - 3y + 3 = 0.$$

Example 7. Find a differential equation for the family indicated by

$$y = C_1x + \frac{C_2}{x}.$$

Solution. From

$$y = C_1x + \frac{C_2}{x} = C_1x + C_2x^{-1} \qquad 5$$

we have

$$y' = C_1 - C_2x^{-2} \qquad 6$$

and

$$y'' = 2C_2x^{-3}. \qquad 7$$

From (7) it follows that

$$C_2 = \frac{x^3}{2}y''. \qquad 8$$

Then from (6) it follows that

$$C_1 = y' + C_2x^{-2} = y' + (\frac{x^3}{2}y'')x^{-2} = y' + \frac{x}{2}y'' \qquad 9$$

Putting the expressions for C_1 and C_2 given in (8) and (9) into (5), we have

$$y = C_1x + C_2x^{-1} = [y' + \frac{x}{2}y'']x + [\frac{x^3}{2}y'']x^{-1}.$$

so

$$y = xy' + \frac{1}{2}x^2y'' + \frac{1}{2}x^2y''.$$

This simplifies to

$$x^2y'' + xy' - y = 0.$$

Additional Examples: See Section 1.2 of the text.

Suggested Problems. Do the odd numbered problems 11- 25 at the end of Section 1.3. Answers are posted on Dr. Walker's web site. There are no problems at the end of Section 1.2.