Section 1.3

Initial Conditions.

A list of the form

$$y(x_0) = k_0, y'(x_0) = k_1, y''(x_0) = k_2, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

is called a set or list of n initial conditions. Such a list of conditions together with an n-th order differential equation is called an n-th order **initial value problem**. It is desirable that initial value problems have unique solutions on some interval.

If the solutions to a differential equation are contained in an n-parameter family, then a list of n initial conditions will often determine the value of each parameter in the family.

Example. It is true that

$$y'' - 4y = 0 \qquad 1$$

if and only if

$$y = C_1 e^{-2x} + C_2 e^{2x} \qquad 2$$

for some pair of numbers C_1 and C_2.

Suppose that y is a solution and y also satisfies the initial conditions

$$y(0) = 1 \text{ and } y'(0) = -1. \qquad 3$$

From

$$y = C_1 e^{-2x} + C_2 e^{2x} \qquad 2$$

and $y(0) = 1$ we have setting $x = 0$ and $y = 1$ that

$$1 = C_1 e^{-2 \cdot 0} + C_2 e^{2 \cdot 0}$$

or

$$C_1 + C_2 = 1. \qquad 4$$

Also from (2) we have

$$y' = -2C_1 e^{-2x} + 2C_2 e^{2x};$$

so using $y'(0) = -1$ we have setting $x = 0$ and $y' = -1$ that

$$-1 = -2C_1 e^{-2 \cdot 0} + 2C_2 e^{2 \cdot 0}$$

or

$$-2C_1 + 2C_2 = -1. \qquad 5$$

$$C_1 + C_2 = 1. \quad 4$$

$$-2C_1 + 2C_2 = -1. \quad 5$$

Solving (4) and (5) (For example, add -2 times (4) to (5) to get C_1 then use (4) to get C_2) we have

$$C_1 = \frac{3}{4} \text{ and } C_2 = \frac{1}{4}.$$

Thus the solution to the initial value problem consisting of (1) and (3) is the function y given by

$$y = \frac{3}{4}e^{-2x} + \frac{1}{4}e^{2x}.$$

Example. Suppose that an object moves along a coordinatized straight line and its displacement from the origin at time t is $y(t)$. Then its velocity $v(t)$ at time t is given by

$$v(t) = y'(t)$$

and its acceleration $a(t)$ is given by

$$a(t) = v'(t)$$

so

$$y''(t) = a(t).$$

Now suppose that a is constant with

$$a(t) = a$$

and that these initial conditions are satisfied:

$$y(0) = y_0 \text{ and } v(0) = y'(0) = v_0$$

Then

$$v'(t) = a.$$

Since

$$\int a dt = at$$

we have

$$v(t) = at + C_1$$

and since $v(0) = v_0$ we have $v_0 = a \cdot 0 + C_1$ yielding $C_1 = v_0$. Thus

$$v(t) = at + v_0.$$

Continuing with

$$v(t) = at + v_0$$

we have

$$y'(t) = at + v_0.$$

Since

$$\int (at + v_0)dt = \frac{1}{2}at^2 + v_0 t$$

we have

$$y(t) = \frac{1}{2}at^2 + v_0 t + C_2.$$

Since $y(0) = y_0$ we have $y_0 = \frac{1}{2}a \bullet 0^2 + v_0 \bullet 0 + C_2$ yielding $C_2 = y_0$. Thus

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0.$$

Example. Find a differential equation for the family indicated by

$$y^2 = Cx^4 - 2. \tag{1}$$

Solution. Differentiating each side of (1) we have

$$2yy' = 4Cx^3 \text{ so } C = \frac{1}{2}\frac{yy'}{x^3}.$$

From (1)

$$y^2 = (\frac{1}{2}\frac{yy'}{x^3})x^4 - 2 \text{ or } 2y^2 = xyy' - 4$$

so

$$y' = \frac{2y^2 + 4}{xy}$$

Example. Find a differential equation for the family indicated by

$$y = Ce^x + \sin x. \tag{1}$$

Solution. From (1) we have

$$y' = Ce^x + \cos x. \tag{2}$$

Subtracting (2) from (1) we have

$$y - y' = \sin x - \cos x$$

or

$$y' - y = \cos x - \sin x.$$

Example. Find a differential equation for the family indicated by

$$y = C_1 e^x + C_2 e^{-2x}. \quad (1)$$

Solution.

$$y' = C_1 e^x - 2C_2 e^{-2x} \quad (2)$$

$$y'' = C_1 e^x + 4C_2 e^{-2x}. \quad (3)$$

Subtracting (2) from (1) yields

$$y - y' = 3C_2 e^{-2x}.$$

Subtracting (3) from (1) yields

$$y - y'' = -3C_2 e^{-2x} \text{ so } y - y'' = -(y - y').$$

Thus

$$y'' + y' - 2y = 0$$

Additional Examples: See Section 1.3 of the text.

Suggested Problems: Do the odd numbered problems 1-25 for Section 1.3 The answers are posted on Dr. Walker's web site.