Engineering Mathematics
Section 1.3

Dr. Philip Walker

Initial Conditions.

A list of the form

$$y(x_0) = k_0, y'(x_0) = k_1, y''(x_0) = k_2, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

is called a set or list of n initial conditions. Such a list of conditions together with an n-th order differential equation is called an n-th order **initial value problem**. It is desirable that initial value problems have unique solutions on some interval.

If the solutions to a differential equation are contained in an n-parameter family, then a list of n initial conditions will often determine the value of each parameter in the family.

Example. It is true that

$$y'' - 4y = 0 \tag{1}$$

on an interval J if and only if

$$y = C_1 e^{-2x} + C_2 e^{2x} \tag{2}$$

on J for some pair of numbers C_1 and C_2.

Suppose that y is a solution and y also satisfies the initial conditions

$$y(0) = 1 \text{ and } y'(0) = -1. \tag{3}$$

From

$$y = C_1 e^{-2x} + C_2 e^{2x} \tag{2}$$

and $y(0) = 1$ we have setting $x = 0$ and $y = 1$ that

$$1 = C_1 e^{-2 \cdot 0} + C_2 e^{2 \cdot 0}$$

or

$$C_1 + C_2 = 1. \tag{4}$$

Also from (2) we have

$$y' = -2C_1 e^{-2x} + 2C_2 e^{2x};$$

so using $y'(0) = -1$ we have setting $x = 0$ and $y' = -1$ that

$$-1 = -2C_1 e^{-2\cdot 0} + 2C_2 e^{2\cdot 0}$$

or

$$-2C_1 + 2C_2 = -1. \qquad (5)$$

$$C_1 + C_2 = 1. \tag{4}$$

$$-2C_1 + 2C_2 = -1. \tag{5}$$

Solving (4) and (5) (For example, add -2 times (4) to (5) to get C_1 then use (4) to get C_2) we have

$$C_1 = \frac{3}{4} \text{ and } C_2 = \frac{1}{4}.$$

Thus the solution to the initial value problem consisting of (1) and (3) is the function y given by

$$y = \frac{3}{4}e^{-2x} + \frac{1}{4}e^{2x}.$$

Example. Suppose that an object moves along a coordinatized straight line and its displacement from the origin at time t is $y(t)$. Then its velocity $v(t)$ at time t is given by

$$v(t) = y'(t)$$

and its acceleration $a(t)$ is given by

$$a(t) = v'(t)$$

so

$$y''(t) = a(t).$$

Suppose that a is constant with

$$a(t) = a$$

and that these initial conditions are satisfied.

$$y(0) = y_0 \text{ and } v(0) = y'(0) = v_0$$

Then

$$v'(t) = a.$$

Since

$$\int a_0 dt = at$$

we have

$$v(t) = at + C_1$$

and since $v(0) = v_0$ we have $v_0 = a \cdot 0 + C_1$ yielding $C_1 = v_0$.Thus

$$v(t) = at + v_0.$$

Continuing with

$$v(t) = at + v_0$$

we have

$$y'(t) = at + v_0.$$

Since

$$\int (at + v_0)dt = \frac{1}{2}at^2 + v_0 t$$

we have

$$y(t) = \frac{1}{2}at^2 + v_0 t + C_2.$$

Since $y(0) = y_0$ we have $y_0 = \frac{1}{2}a \cdot 0^2 + v_0 \cdot 0 + C_2$ yielding $C_2 = y_0$.Thus

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0.$$

Additional Examples: See Section 1.3 of the text.

Suggested Problems: Do the odd numbered problems 1-25 for Section 1.3 The answers are posted on Dr. Walker's web site.