Section 2.1
First Order Linear Differential Equations

Definition *A* ***first order linear*** *differential equation is one that is equivalent to one of the form*

$$y' + p(x)y = f(x)$$

Suppose that each of p and f is continuous on an interval J. To solve the first order linear differential equation

$$y' + p(x)y = f(x)$$

begin by finding an anti-derivative h of p.

$$h(x) = \int p(x)dx$$

Leave off the $+C$. Note that

$$h'(x) = p(x).$$

$$y' + p(x)y = f(x) \qquad 1$$

Multiply each side of (1) by

$$e^{h(x)}.$$

This function is called the **integrating factor**. The result is

$$y'e^{h(x)} + p(x)e^{h(x)}y = f(x)e^{h(x)}. \qquad 2$$

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Using the product rule, it follows that the left side of (2) is the derivative of $ye^{h(x)}$. Thus

$$(ye^{h(x)})' = f(x)e^{h(x)}. \qquad 3$$

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Integrating, it follows that

$$ye^{h(x)} = Q(x) + C \qquad 4$$

where Q is an antiderivative of the right side of (3).

$$Q(x) = \int f(x)e^{h(x)}dx \tag{5}$$

$$ye^{h(x)} = Q(x) + C \tag{4}$$

Thus

$$y = Ce^{-h(x)} + e^{-h(x)}Q(x). \tag{6}$$

If y is given by (6), differentiation shows that y is a solution to (1). The function y is a solution to (1) if and only if y is given by (6) for some constant C.

Example. Find all solutions (or find the general solution) to

$$y' - 2xy = x. \tag{7}$$

Solution: The integrating factor is

$$e^{\int(-2x)dx} = e^{-x^2}.$$

Multiplying each side of (7) by it gives

$$y'e^{-x^2} - 2xe^{-x^2}y = xe^{-x^2}$$

which is equivalent to

$$(ye^{-x^2})' = xe^{-x^2}.$$

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Noting that

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}$$

it follows that

$$ye^{-x^2} = C - \frac{1}{2}e^{-x^2}$$

so

$$y = Ce^{x^2} - \frac{1}{2}.$$

Example. Find the solution to

$$y' - 2xy = x \text{ and } y(0) = 0.$$

Solution: It follows from the last example that

$$y(x) = Ce^{x^2} - \frac{1}{2}$$

for some constant C. Since $y(0) = 0$ we have

$$Ce^{0^2} - \frac{1}{2} = 0$$

So

$$C - \frac{1}{2} = 0$$

yielding

$$C = \frac{1}{2}.$$

Thus the solution to the IVP is given by

$$y = \frac{1}{2}e^{x^2} - \frac{1}{2}.$$

Note. The integration by parts formula is

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

In the next example we will need

$$\int \ln x dx.$$

$$\int \ln x dx = \int (\ln x)(1)dx$$

Let u and v be such that $u(x) = \ln x$ and $v'(x) = 1$ so $u'(x) = \frac{1}{x}$ and $v(x) = \int 1dx = x$.

$$\int \ln x dx = \int (\ln x)(1)dx = (\ln x)(x) - \int (\frac{1}{x})(x)dx = x\ln x - \int 1dx$$

$$\int \ln x dx = x\ln x - x$$

Example. Find all solutions (or find the general solution) to

$xy' + 3y = \frac{\ln x}{x^2}$ on the set of positive numbers.

Solution. First divide each side of the equation by x to put it in the standard form for a first order linear equation.

$$y' + \frac{3}{x}y = \frac{\ln x}{x^3}$$ 8

Next get the integrating factor.

$$\int \frac{3}{x}dx = 3\ln x = \ln x^3.$$

Remember that

$e^{\ln z} = z$ for every positive number z

The integrating factor is

$$e^{\ln x^3} = x^3.$$

Multiplying each side of (8) by the integrating factor produces

$$x^3y' + \frac{3}{x}x^3y = \frac{\ln x}{x^3}x^3$$

or

$$x^3y' + 3x^2y = \ln x \text{ or } (x^3y)' = \ln x.$$

$$\int \ln x dx = x\ln x - x$$

Thus

$$x^3y = x\ln x - x + C$$

so

$$y = \frac{C}{x^3} + \frac{\ln x}{x^2} - \frac{1}{x^2}$$

Example. Find all solutions (or find the general solution) to

$xy' + 2y = \dfrac{2}{\sqrt{x^2-1}} - 2x^2$ on the set of numbers greater than one.

Solution. First divide each side by x to produce

$$y' + \frac{2}{x}y = \frac{2}{x\sqrt{x^2-1}} - 2x. \qquad 9$$

The integrating factor is

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Multiplying each side of (9) by the integrating factor, we have

$$x^2y' + 2xy = \frac{2x}{\sqrt{x^2-1}} - 2x^3$$

or

$$(x^2y)' = \frac{2x}{\sqrt{x^2-1}} - 2x^3.$$

$$\int\left(\frac{2x}{\sqrt{x^2-1}} - 2x^3\right)dx = \int\left((x^2-1)^{-\frac{1}{2}}(2x) - 2x^3\right)dx = 2(x^2-1)^{\frac{1}{2}} - \frac{1}{2}x^4$$

so

$$(x^2y) = 2(x^2-1)^{\frac{1}{2}} - \frac{1}{2}x^4 + C$$

and

$$y = \frac{C}{x^2} + \frac{2(x^2-1)^{\frac{1}{2}}}{x^2} - \frac{1}{2}x^2$$

or

$$y = \frac{C}{x^2} + \frac{2\sqrt{(x^2-1)}}{x^2} - \frac{1}{2}x^2$$

Example. Find the solution to

$y' + \cot(x)y = 2\cos x$ on the set of numbers between 0 and π and $y(\frac{\pi}{2}) = 3.$

Solution. The integrating factor is

$$e^{\int \cot x dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln \sin x} = \sin x$$

and multiplying each side of the DE by it produces

$y' \sin x + \cos(x)y = 2\sin x \cos x$ or $(y \sin x)' = 2\sin x \cos x.$

$$\int 2\sin x \cos x dx = \sin^2 x$$

so

$$y \sin x = \sin^2 x + C$$

and

$$y = \frac{C}{\sin x} + \sin x$$

Since

$y(\frac{\pi}{2}) = 3$ we have $3 = \frac{C}{\sin\frac{\pi}{2}} + \sin x \frac{\pi}{2}$ or $3 = \frac{C}{1} + 1$

so

$$C = 2$$

and

$$y = \frac{2}{\sin x} + \sin x.$$

or

$$y = 2\csc x + \sin x$$

Note. Suppose that the right side of a first order linear differential equation in standard form is zero so that

$$y' + p(x)y = 0.$$

Let $h(x) = \int p(x)dx$ and multiply each side of the DE by the integrating factor $e^{h(x)}$ to get

$$y'e^{h(x)} + p(x)e^{h(x)}y = 0 \text{ or } (ye^{h(x)})' = 0 \text{ so } ye^{h(x)} = C$$

or

$$y = Ce^{-h(x)} = Ce^{-\int p(x)dx} \qquad 11$$

for some constant C. Conversely, if y is given by (11) for some constant C, then y is an solution to (10).

If p is continuous on an interval J, x_0 is a number in J and

$$h(x) = \int_{x_0}^{x} p(t)dt$$

(h is the specific anti-derivative of p such that $h(x_0) = 0$) then

$$y(x_0) = Ce^{-\int_{x_0}^{x_0} p(t)dt} = Ce^0 = C$$

so

$$y(x) = y(x_0)e^{-\int_{x_0}^{x} p(t)dt}.$$

Consequently, if $y(x_0) \neq 0$ for some x_0 in J, then $y(x) \neq 0$ for all x in J, and if $y(x_0) = 0$ for some x_0 in J, then $y(x) = 0$ for all x in J.

Definition. Saying that L is a linear operator acting on a collection of functions S means if y is in S and c is a number then cy is in S and

$$L[cy] = cL[y],$$

and if each of y_1 and y_2 is in S then $y_1 + y_2$ is in S and

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$

Example. Differentiation acting on the differentiable functions defined on an interval is a linear operator. If

$$L[y] = y'$$

then

$$L[cy] = (cy)' = cy' = cL[y]$$

and

$$L[y_1 + y_2] = (y_1 + y_2)' = y_1' + y_2' = L[y_1] + L[y_2].$$

Definition. Saying that L is a first order linear differential operator over an interval J

means that there is a function p with domain J such that

$$L[y] = y' + p(x)y$$

whenever y is a differentiable function with domain J.

Note that in standard form, a first order linear differential equation is of the form

$$L[y] = f$$

where L is as above.

Theorem. **If** p is a function defined on an interval J and

$$L[y] = y' + p(x)y$$

whenever y is a differentiable function defined of J, then L is a linear operator.

Proof.

$$L[cy] = (cy)' + p(x)(cy) = cy' + cp(x) = c(y' + p(x)y) = cL[y]$$

and

$$\begin{aligned} L[y_1 + y_2] &= (y_1 + y_2)' + p(x)(y_1 + y_2) = y_1' + y_2' + p(x)y_1 + p(x)y_2 \\ &= y_1' + p(x)y_1 + y_2' + p(x)y_2 = L[y_1] + L[y_2] \end{aligned}$$

whenever each of y, y_1, y_2 is a differentiable function defined on J and c is a number.

Example. Suppose that the operator L is given by

$$L[y] = y' + \frac{2}{x}y$$

whenever y is a differentiable function defined on the positive numbers. Then

$$L[2x^3 + x] = (6x^2 + 1) + \frac{2}{x}(2x^3 + x) = 10x^2 + 3,$$

$$L[e^{2x}] = 2e^{2x} + \frac{2}{x}e^{2x} = (2 + \frac{2}{x})e^{2x}$$

and

$$L[x^2] = 2x + \frac{2}{x} \cdot x^2 = 4x.$$

Definition. When each of $y_1$and $y_2$is a function defined on a set J and each of c_1 and c_2 is a number,

$$c_1y_1 + c_2y_2$$

is called a **linear combination** of y_1 and y_2.

Theorem. If L is a linear operator acting on a collection of functions S, each of y_1 and y_2 is in S and each of c_1 and c_2 is a number then

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

Additional Examples: See Section 2.1 of the text.

Suggested Problems. Do the odd numbered problems for Section 2.1. The answers are posted on Dr. Walker's web site.