Section 2.2 Examples

3. $y' = xy^2 + x$: $y' = x(y^2+1)$: $\frac{y'}{y^2+1} = x$

$\arctan y = \frac{1}{2}x^2 + C$: $\boxed{y = \tan(\frac{1}{2}x^2 + C)}$

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4. $y' = \frac{xy^2 + 4x}{2y}$: $y' = \frac{x(y^2+4)}{2y}$: $\frac{2yy'}{y^2+4} = x$

$\int \frac{2y\,dy}{y^2+4} = \int x\,dx$: $\ln(y^2+4) = \frac{1}{2}x^2 + C$

$y^2 + 4 = Ce^{\frac{1}{2}x^2}$ or $\boxed{y^2 = Ce^{\frac{1}{2}x^2} - 4}$

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5. $\frac{1}{x} \cdot y' = e^x\sqrt{y+1}$: $\frac{y'}{\sqrt{y+1}} = xe^x$

$\int (y+1)^{-1/2}(1)\,dy = \int xe^x\,dx$

$\int \underset{u(x)}{x}\ \underset{v'(x)}{e^x}\,dx = xe^x - \int e^x\,dx = (x-1)e^x$

$2(y+1)^{1/2} = (x-1)e^x + C$

$\sqrt{y+1} = \frac{1}{2}(x-1)e^x + C$

$y = \left(\frac{1}{2}(x-1)e^x + C\right)^2 - 1$

$y = -1$ is a singular solution.

6. $y' - xy^2 = -x$: $y' = xy^2 - x = x(y^2-1)$

If $y \neq 1$ and $y \neq -1$ $\Rightarrow$ $\frac{y'}{y^2-1} = x$: $\int \frac{dy}{y^2-1} = \int x\,dx + C$

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} = \frac{A(y+1)+B(y-1)}{(y-1)(y+1)}$$

$1 = A(y+1) + B(y-1)$

$y = 1$: $1 = A \cdot 2$; $A = \frac{1}{2}$

$y = -1$: $1 = B(-1-1)$: $B = -\frac{1}{2}$

$$\frac{1}{y^2-1} = \frac{1/2}{y-1} - \frac{1/2}{y+1}$$

$$\int \left[\frac{1/2}{y-1} - \frac{1/2}{y+1}\right] dy = \int x\,dx + C$$

$$\frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = \frac{1}{2}x^2 + C$$

$y > 1$

$$\ln|y-1| - \ln|y+1| = x^2 + C$$

remove || $\ln(y-1) - \ln(y+1) = x^2 + C$

$$\ln\left(\frac{y-1}{y+1}\right) = x^2 + C$$

$y - 1 > 0$

$y + 1 > y - 1 > 0$

new C

$$\frac{y-1}{y+1} = e^{x^2+C} = Ce^{x^2}$$

$$y - 1 = Ce^{x^2}(y+1)$$

$$y(1 - Ce^{x^2}) = Ce^{x^2} + 1$$

$$\boxed{y = \frac{1+Ce^{x^2}}{1-Ce^{x^2}}}$$

$|z| = \begin{cases} z \\ -z \end{cases}$

$-1<y<1$: $\ln|y-1| - \ln|y+1| = x^2+C$

$0<y+1$

$y-1<0$

$\ln(1-y) - \ln(1+y) = x^2+C$

$\ln \dfrac{1-y}{1+y} = x^2+C$

$\dfrac{1-y}{1+y} = Ce^{x^2}$

$1-y = ce^{x^2} + yce^{x^2}$

$1-ce^{x^2} = y(1+ce^{x^2})$

$$\boxed{y = \frac{1-ce^{x^2}}{1+ce^{x^2}}}$$

$y<-1$

$y+1<0$

$y-1<y+1<0$

$\ln|y-1| - \ln|y+1| = x^2+C$

$\ln(1-y) - \ln(-y-1) = x^2+C$

$\ln\left(\dfrac{1-y}{-y-1}\right) = x^2+C$

$\ln\left(\dfrac{y-1}{y+1}\right) = x^2+C$

$\dfrac{y-1}{y+1} = Ce^{x^2}$

$y-1 = yce^{x^2} + Ce^{x^2}$

$y(1-ce^{x^2}) = 1+ce^{x^2}$

$$\boxed{y = \frac{1+ce^{x^2}}{1-ce^{x^2}}}$$

$y=1$ and $y=-1$ are singular solutions