Section 2.2

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Separable Differential Equations

Definition. A differential equation that is equivalent to one of the form

$$f(y)y' = g(x) \qquad 1$$

is said to be **separable**. When in this form, It can be solved by integrating each side. The result is

$$F(y) = G(x) + C \qquad 2$$

where F is an anti-derivative of f and G is an anti-derivative of g.

$$F(y) = \int f(y)dy \text{ and } G(x) = \int g(x)dx$$

Equation (2) gives an implicit description of the solutions. It may or may not be possible to solve it for y explicitly.

Although lacking in mathematical rigor, a helpful notational device is to replace y' in

$$f(y)y' = g(x) \qquad 1$$

with

$$\frac{dy}{dx}$$

multiply each side of the resulting equation by dx resulting in

$$f(y)dy = g(x)dx$$

and supply an integral sign to each side resulting in

$$\int f(y)dy = \int g(x)dx + C.$$

Example 1. Solve

$$y' = \frac{y^2+1}{xy+y}.$$

Solution. The given d.e. is equivalent to

$$y' = \frac{y^2+1}{y(x+1)}$$

which is equivalent to

$$\frac{y}{y^2+1}y' = \frac{1}{x+1}.$$

This is of the form given in equation (1).

Next we have

$$\frac{y}{y^2+1} \cdot \frac{dy}{dx} = \frac{1}{x+1}$$

Then

$$\int \frac{y}{y^2+1} dy = \int \frac{1}{x+1} dx + C$$

so

$$\frac{1}{2} \int \frac{2y}{y^2+1} dy = \int \frac{1}{x+1} dx + C.$$

For all x in some interval. Integrating each side we have

$$\frac{1}{2} \ln(y^2+1) = \ln|x+1| + C.$$

This gives an implicit description of the solutions. We will try to solve for y.

$$\ln(y^2+1) = 2\ln|x+1| + C$$

or

$$\ln(y^2+1) = \ln((x+1)^2) + C.$$

Exponentiating, this becomes

$$e^{\ln(y^2+1)} = e^{\ln((x+1)^2)+C.}.$$

Note that

$$e^{\ln((x+1)^2)+C.} = e^{\ln((x+1)^2)} e^C$$

Remember that $e^{\ln z} = z$ for all $z > 0$. Thus

$$y^2 + 1 = (x+1)^2 \cdot C.$$

So

$$y^2 = C(x+1)^2 - 1$$

or

$$y = \pm\sqrt{C(x+1)^2 - 1.}$$

An initial condition is needed to determine whether to use $+$ or $-$.

Example 2. Solve (find the general solution to)

$$y' = 3x(1+y^2).$$

Solution. The given d.e. is equivalent to

$$\frac{1}{1+y^2}y' = 3x$$

This is of the form given in Equation (1). Continuing, we have

$$\int \frac{1}{1+y^2}dy = \int 3x dx + C$$

so

$$\arctan y = \frac{3}{2}x^2 + C.$$

Thus

$$y = \tan\left(\frac{3}{2}x^2 + C\right)$$

Additional Examples: See Additional Examples for Section 2.2 posted on Dr. Walker's web site, Section 2.2 of the text, and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for section 2.2. The answers are posted on Dr. Walker's web site.