Section 2.3

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Two More First Order Differential Equations

Definition. A differential equation that is equivalent to one of the form

$$y' + p(x)y = q(x)y^r$$

where $r \neq 0$ and $r \neq 1$ is said to be a **Bernoulli** differential equation.

When the equation is in this form, it can be solved by dividing each side by y^r and letting

$$v = y^{1-r}.$$

The result is a first order linear differential equation for v. Find v, then noting that

$$y = v^{\frac{1}{1-r}}$$

find y.

Example. Find all solutions to

$$y' + \frac{1}{x}y = 3x^2y^2 \text{ on intervals of positive numbers.}$$

Solution. Multiply each side by y^{-2} to get

$$y'y^{-2} + \frac{1}{x}y^{-1} = 3x^2.$$

Let

$$v = y^{-1} \text{ then } v' = -y^{-2}y'$$

so

$$-v' + \frac{1}{x}v = 3x^2.$$

or

$$v' - \frac{1}{x}v = -3x^2.$$

The integrating factor for this first order linear d.e. is

$$e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

so

$$\frac{1}{x}v' - \frac{1}{x^2}v = -3x$$

or

$$(\frac{1}{x}v)' = -3x.$$

Integrating we have

$$\frac{1}{x}v = -\frac{3}{2}x^2 + C$$

so

$$v = Cx - \frac{3}{2}x^3.$$

Since

$$y = \frac{1}{v}$$

it follows that

$$y = \frac{1}{Cx - \frac{3}{2}x^3}$$

or

$$y = \frac{2}{Cx - 3x^3}.$$

The C in the last equation is twice the C in the previous equation. The constant function with value zero is a singular solution.

Saying that a differential equation of the form

$$y' = f(x,y)$$

is **homogeneous** in x and y means that

$$f(\lambda x, \lambda y) = f(x,y)$$

for all λ in some interval. To solve such an equation, let v be such that

$$y = xv.$$

The result will be a separable equation for v. Solve for v then find y.

Note: In this section of the text these equations are simply called homogeneous. The term homogeneous usually refers to a linear equation where the right side is zero. That is the usage you find beginning in Chapter 3.

Example. Find a parameterized family of solutions (or find the general solution) to

$$y' = \frac{x^2 + y^2}{2xy} \text{ on intervals of positive numbers.}$$

Solution. The differential equation is of the form

$$y' = f(x,y)$$

where

$$f(x,y) = \frac{x^2+y^2}{2xy}.$$

Note that

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{2\lambda x \lambda y} = \frac{\lambda^2(x^2+y^2)}{\lambda^2 2xy} = f(x,y).$$

So the differential equation is of homogeneous degree.

Ley v be such that $xv = y$. Then from the differential equation we have

$$v + xv' = \frac{x^2 + (xv)^2}{2x(xv)}$$

so

$$xv' = \frac{1+v^2}{2v} - v$$

or

$$xv' = \frac{1+v^2}{2v} - \frac{2v^2}{2v}$$

or

$$xv' = \frac{1-v^2}{2v}.$$

From this we get

$$\frac{2vv'}{1-v^2} = \frac{1}{x}$$

so

$$-\int \frac{-2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx + C_1.$$

Thus

$$-\ln|1-v^2| = \ln x + C_1$$

so

$$\ln|1-v^2| = \ln\frac{1}{x} + C_2$$

so

$$e^{\ln|1-v^2|} = e^{\ln\frac{1}{x}+C_2}$$

so

$$|1-v^2| = C_3\frac{1}{x}$$

where $C_3 = e^{c_2}$. Since $xv = y$ we have

$$\left|1 - \frac{y^2}{x^2}\right| = C_3\frac{1}{x}$$

or

$$|x^2 - y^2| = C_3 x.$$

For solutions where $x^2 \geq y^2$ this becomes

$$y^2 = Cx + x^2$$

where $C = -C_3$. For solutions where $x^2 < y^2$this becomes

$$y^2 = Cx + x^2$$

where $C = C_3$.

Additional Examples. See those in Section 2.3 of the text and those presented on the board in class.

Suggested problems for Section 2.3. Do the odd numbers 1 through 25.