Engineering Mathematics

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Section 2.4.1
Orthogonal Trajectories

Orthogonal means perpendicular.

Lines in the plane are perpendicular or orthogonal if and only if one is vertical and the other horizontal or the product of their slopes is -1.

Smooth curves in the plane are orthogonal at a point of intersection if and only if their tangent lines are orthogonal at that point.

Suppose that $\mathcal{F}$ is a one-parameter family of smooth curves in the plane. Saying that $\mathcal{G}$ is the family of **orthogonal trajectories** for $\mathcal{F}$ means that $\mathcal{G}$ is a one-parameter family of smooth curves in the plane and if f is in $\mathcal{F}$, g is in $\mathcal{G}$ and f and g intersect at a point, they are orthogonal at that point.

Example. The straight lines through the origin form the orthogonal trajectories for the circles centered at the origin. In plane geometry, we learn that a radius and tangent that meet at a point on a circle are perpendicular.

Procedure. To find the orthogonal trajectories for a given family, find the differential equation for the family, replace y' in that equation with $-1/y'$, and solve the resulting differential equation. The solutions will form the family of orthogonal trajectories.

Example. Find the orthogonal trajectories for the family of parabolas indicated by

$$x = Cy^2. \tag{1}$$

Solution. From (1) we have

$$1 = 2Cyy'. \tag{2}$$

Solving (1) for C to get

$$C = \frac{x}{y^2}$$

and putting this value of C into (2) we have

$$1 = 2\frac{x}{y^2}yy'$$

so

$$1 = 2\frac{x}{y}y'. \tag{3}$$

This is the d.e. for the given family.

$$1 = 2\frac{x}{y}y'. \tag{3}$$

Replacing y' with $-1/y'$ we have

$$1 = -2\frac{x}{yy'}$$

or

$$yy' = -2x \tag{4}$$

This is the d.e. for the orthogonal trajectories. It is separable. Solving it by integration we have

$$\frac{1}{2}y^2 = -x^2 + C$$

or

$$\frac{x^2}{1} + \frac{y^2}{2} = C \tag{5}$$

(5) gives the orthogonal trajectories. It is a family of ellipses.

Additional Examples: See Section 2.4.1 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for section 2.4.1. The answers are posted on Dr. Walker's web site.