Engineering Mathematics

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Section 2.4.2
Exponential Growth and Decay

The differential equation

$$A'(t) = rA(t) \text{ for all } t \text{ in } J \tag{1}$$

indicates that the rate of change in A at time t is proportional to the amount $A(t)$. The interval J is usually $[0, \infty)$ and we will take it to be that. The equation can be solved by writing it in the equivalent form

$$A'(t) - rA(t) = 0$$

which is a first order linear differential equation. It can also be solved by writing it in the equivalent form (assuming that $A(t) \neq 0$)

$$\frac{A'(t)}{A(t)} = r$$

which is a separable differential equation.

Either way, we find that A is a solution to

$$A'(t) = rA(t) \text{ for all } t \text{ in } J \tag{1}$$

if and only if

$$A(t) = ce^{rt}$$

for some number C and all t in J. Setting $t = 0$, it follows that

$$A(0) = C$$

so

$$A(t) = A(0)e^{rt} \text{ for all } t \text{ in } J. \tag{2}$$

Equation (2) gives the solution to (1) in terms of $A(0)$ which is called the initial amount.

GROWTH

When $r > 0$, A is said to exhibit exponential growth and r is called the growth constant. In this case, the doubling time is the number D such that

$$A(D) = 2A(0).$$

From (2), letting $t = D$, we see that

$$2A(0) = A(0)e^{rD}.$$

Thus (assuming that $A(0) \neq 0$)

$$2 = e^{rD}$$

so

$$\ln 2 = rD$$

so

$$D = \frac{\ln 2}{r}$$

and

$$r = \frac{\ln 2}{D}.$$

The doubling time also has the following property. When t_0 is any number in J,

$$A(t_0 + D) = 2A(t_0).$$

To see this note that

$$\begin{aligned} A(t_0 + D) &= A(0)e^{r(t_0+D)} = A(0)e^{rt_0}e^{rD} \\ &= A(0)e^{rD}e^{rt_0} = 2A(0)e^{rt_0} = 2A(t_0) \end{aligned}$$

DECAY

When $r < 0$, A is said to exhibit exponential decay and $-r$ is called the decay constant. Let

$$k = -r \text{ so that } r = -k.$$

Equation (2) becomes

$$A(t) = A(0)e^{-kt} \text{ for all } t \text{ in } J. \tag{3}$$

In this case, the half-life is the number H

$$A(H) = \frac{1}{2}A(0).$$

From (3), letting $t = H$,we see that

$$\frac{1}{2}A(0) = A(0)e^{-kH}.$$

Thus (assuming that $A(0) \neq 0$)

$$\frac{1}{2} = e^{-kH}$$

so

$$\ln\frac{1}{2} = -kH$$

so

$$-\ln 2 = -kH$$

so

$$H = \frac{\ln 2}{k}$$

and

$$k = \frac{\ln 2}{H}.$$

The half-life also has the following property. When t_0 is any number in J,

$$A(t_0 + H) = \frac{1}{2}A(t_0).$$

To see this note that

$$\begin{aligned} A(t_0 + H) &= A(0)e^{-k(t_0+H)} = A(0)e^{-kt_0}e^{-kH} = A(0)e^{-kH}e^{-kt_0} \\ &= \frac{1}{2}A(0)e^{-kt_0} = \frac{1}{2}A(t_0) \end{aligned}$$

EXAMPLES

What is the half-life of a radioactive substance if it takes 5 years for one-third of the material to decay?

Solution:

$$A(t) = A(0)e^{-kt}$$

so

$$A(5) = A(0)e^{-5k}$$

After 5 years two-thirds of the material remains so

$$A(5) = \frac{2}{3}A(0).$$

Thus

$$\frac{2}{3}A(0) = A(0)e^{-5k}.$$

Dividing each side by $A(0)$ then taking ln of each side we have

$$\ln\frac{2}{3} = -5k.$$

Thus

$$k = -\frac{\ln(2/3)}{5} = \frac{\ln 3 - \ln 2}{5}.$$

Since the half-life is $\dfrac{\ln 2}{k}$, it follows that

$$H = \frac{5\ln 2}{\ln 3 - \ln 2} \approx 8.548 \text{ years}$$

Additional Examples: See Section 2.4.2 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for section 2.4.2. The answers are posted on Dr. Walker's web site.