Notes on Section 2.4.3
Newton's Law of Cooling

Philip W. Walker

Newton's law of cooling states that the rate of change in the temperature of an object is proportional to the difference in the temperature of the object and the temperature of the surrounding medium.Thus

$$u'(t) = -k(u(t) - \sigma) \text{ for } t \geq 0 \tag{1}$$

where $u(t)$ is the temperature of the object at time t, σ is the temperature of the surrounding medium, and k is a positive constant.

The equation (1) is equivalent to

$$u'(t) + ku(t) = k\sigma \text{ for } t \geq 0 \tag{2}$$

which is a first order linear equation and also to

$$\frac{u'(t)}{u(t) - \sigma} = -k \text{ for } t \geq 0 \tag{3}$$

which is separable. Solving (2) or (3), you should find that u is a solution to (1) if and only if

$$u(t) = \sigma + ce^{-kt}$$

for some number c and all $t \geq 0$. Using this, you should find that the solution to (1) satisfying the initial condition

$$u(0) = u_0$$

is given by

$$u(t) = \sigma + (u_0 - \sigma)e^{-kt} \text{ for } t \geq 0.$$

Note that

$$\lim_{t \to \infty} u(t) = \sigma.$$

Example. A metal ball at room temperature $20°C$ is dropped into a container of boiling water at $100°C$. Given that the temperature of the ball increases $2°$in 2 seconds, find the temperature $u(t)$ t seconds after the ball is dropped into the boiling water.

Solution.

$$u(t) = 100 + (20 - 100)e^{-kt} \tag{4}$$

and $u(2) = 20 + 2$ so

$$22 = 100 - 80e^{-k \cdot 2}.$$

From this we get

$$\frac{78}{80} = e^{-2k}.$$

Taking ln of each side we find that

$$k = -\frac{1}{2}\ln(\frac{78}{80}) = \ln(\frac{80}{78})^{\frac{1}{2}}.$$

Using the fact that

$$-t\ln(\frac{80}{78})^{\frac{1}{2}} = \ln(\frac{78}{80})^{\frac{t}{2}}$$

it follows from (4) that

$$u(t) = 100 - 80(\frac{78}{80})^{\frac{t}{2}}$$

for $t \geq 0$.

Example. See the example on pages 49 and 50 of the text.

Suggested Problems. 1, 3, and 5 in Exercises 2.4.3 on pages 50 and 51 of the text.