Notes on Section 2.4.5
Mixing Problems

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A fluid containing a substance (e.g. a salt water solution) with concentration k_1 flows into a mixing container at a rate of R_1 and is mixed with the fluid already there. The mixed fluid is pumped out at a rate R_2. Let $A(t)$ be the amount of the substance in the mixing container at time t and let $V(t)$ be the volume of the fluid in the mixing container at time t.

The rate of change in A is the rate that the substance is flowing into the container minus the rate that the substance is flowing out of the container. The inflow rate of the substance is k_1R_1. The concentration of the substance in the container at time t is $\frac{A(t)}{V(t)}$. This is also the concentration of the substance in the fluid that is being pumped out. Thus the outflow rate of the substance is $\frac{A(t)}{V(t)}R_2$.

Putting everything together, we have

$$A'(t) = k_1R_1 - \frac{A(t)}{V(t)}R_2 \text{ for } t \geq 0. \tag{1}$$

Note that V is constant when $R_1 = R_2$ and

$$V(t) = V(0) + (R_1 - R_2)t$$

when $R_1 \neq R_2$.

Note that (1) is equivalent to

$$A'(t) + \frac{R_2}{V(t)}A(t) = k_1R_1 \text{ for } t \geq 0 \tag{2}$$

which is a first order linear differential equation.

Example. A 100 gallon tank is initially full of water. At time $t = 0$ a 20% hydrochloric acid solution begins to flow into the tank at a rate of 2 gallons/minute. The well-mixed solution in the tank is pumped out at the same rate. Find the amount $A(t)$ (in gallons) of acid in the tank at time t.

Solution. Note that V is constant and has the value 100. Thus (2) becomes

$$A'(t) + \frac{2}{100}A(t) = 0.20 \cdot 2$$

which becomes

$$A'(t) + \frac{1}{50}A(t) = 0.40.$$

Using the integrating factor $e^{t/50}$ we have

$$A'(t)e^{t/50} + \frac{1}{50}e^{t/50}A(t) = 0.40e^{t/50}.$$

Defining B by

$$B(t) = A(t)e^{t/50}$$

the last differential equation becomes

$$B'(t) = 0.40e^{t/50}.$$

$$\int 0.40e^{t/50}dt = 20e^{t/50}$$

so

$$B(t) = C + 20e^{t/50}.$$

Thus

$$A(t) = Ce^{-t/50} + 20$$

for some number C and all $t \geq 0$. Since the mixing tank contains only water when $t = 0$, we have

$$A(0) = 0.$$

Thus $C = -20$ and

$$A(t) = 20(1 - e^{-t/50}) \text{ for } t \geq 0.$$

Example. See the example on pages 52 and 53 of the text.

Suggested Problems. Problems 1, 3, and 5, in Exercises 2.4.5 on pages 53 and 54 of the text.