Notes on Section 2.4.6
The Logistic Equation

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The differential equation

$$y'(t) = ky(t)\left(M - y(t)\right) \text{ for } t \geq 0 \tag{1}$$

where each of k and M is a positive number is known as the logistic equation. The equation has applications in mathematical biology. For example, $y(t)$ could be the number of people in a population of M people who have been infected by a certain disease by time t. The number M is called the carrying capacity and k is called the intrinsic growth rate.

Equation (1) is equivalent to

$$y'(t) - kMy(t) = -k\left(y(t)\right)^2 \text{ for } t \geq 0$$

which is a Bernoulli differential equation. It is also equivalent to

$$\frac{y'(t)}{y(t)(M - y(t))} = k \text{ for } t \geq 0 \tag{2}$$

which is separable. We will find the solutions to (1) by solving (2). Using partial fractions

$$\frac{1}{z(z-r)} = \frac{A}{z} + \frac{B}{z-r}$$

on the left side of (2) we have

$$\int \left(\frac{1/M}{y(t)} y'(t) + \frac{1/M}{M - y(t)} y'(t) \right) dt = \int k dt$$

In the applications of interest it will be the case that $0 < y(t) < M$ so

$$\frac{1}{M} \ln y(t) - \frac{1}{M} \ln(M - y(t)) = kt + C_1$$

so

$$\ln \frac{y(t)}{M - y(t)} = Mkt + C_2.$$

Exponentiating and noting that the additive constant becomes a multiplicative one,

$$\frac{y(t)}{M - y(t)} = Ce^{Mkt}$$

so

$$y(t) = \frac{CM}{C + e^{-Mkt}}.$$

If

$$y(0) = R$$

where $0 < R < M$ then

$$y(t) = \frac{MR}{R + (M - R)\, e^{-Mkt}} \text{ for } t \geq 0.$$

Example. See the example on page 56 of the text.

Suggested Problems. Problems 1,3, and 5 in Exercises 2.4.6 on pages 56 and 57 of the text.