Notes on Section 2.4.7

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The following are solutions and solution suggestions for some of the problems in this subsection.

1. A disease is infecting a colony of 1000 penguins living on a remote island Let $P(t)$ be the number of sick penguins t days after the outbreak. Suppose that 50 penguins had the disease initially, and suppose that the disease is spreading at a rate proportional to the product of the time elapsed and the number of penguins who do not have the disease.

   (a) The model is
   $$P' = kt(1000 - P)$$
   and
   $$P(0) = 50.$$

   (b) The differential equation is separable.
   $$\frac{P'}{1000 - P} = kt$$
   $$\int \frac{dP}{1000 - P} = \int ktdt$$
   $$-\ln(1000 - P) = \frac{1}{2}kt^2 + C$$
   $$1000 - P = Ce^{-\frac{1}{2}kt^2}$$
   $$P = 1000 - Ce^{-\frac{1}{2}kt^2}$$
   is the general solution to the differential equation.

   (c) The initial condition is
   $$P(0) = 50$$
   so
   $$50 = 1000 - C$$
   $$C = 950$$
   $$P = 1000 - 950e^{-\frac{1}{2}kt^2}$$
   is the solution to the initial value problem. More information is needed to determine the value of k.

2. A disease is infecting a herd of 100 cows. Let $P(t)$ be the number of sick cows t days after the outbreak. Suppose that 15 cows had the disease initially, and suppose that the disease is spreading at a rate proportional to the number of cows who do not have the disease.

   (a) The model is

   $$P' = k(100 - P)$$

   and

   $$P(0) = 15.$$

   (b) The differential equation is separable.

   $$\frac{p'}{100 - P} = k$$

   $$\int \frac{dP}{100 - P} = \int kdt + C$$

   $$-\int \frac{(-1)dP}{100 - P} = \int kdt + C$$

   $$-\ln(100 - P) = kt + C$$

   $$\ln(100 - P) = -kt + C$$

   $$100 - P = Ce^{-kt}$$

   $$P = 100 - Ce^{-kt}$$

   is the general solution.

   (c)

   $$P(0) = 15$$

   so

   $$15 = 100 - Ce^{-k \cdot 0} = 100 - C.$$

   Thus

   $$C = 85$$

   and

   $$P(t) = 100 - 15e^{-kt}$$

   This is the solution to the initial value problem. More information is needed to determine the value of k.

3. A 1000-gallon tank, initially full of water, develops a leak at the bottom. Given that 200 gallons of water leaks out in the first 10 minutes, find the amount of water, $A(t)$ in the tank t minutes after the leak develops if water drains off at a rate proportional to the product of the time elapsed and the amount of water present.

Solution.

$$A' = ktA$$
$$A(0) = 1000$$
$$A(10) = 800$$

Solving the d.e. we have

$$\frac{A'}{A} = kt$$
$$\int \frac{dA}{A} = \int ktdt + C$$
$$\ln A = \frac{1}{2}kt^2 + C$$
$$A = Ce^{kt^2/2}$$

applying $A(0) = 1000$ we have

$$1000 = Ce^0 \text{ so } C = 1000 \text{ and}$$
$$A = 1000e^{kt^2/2}. \qquad (*)$$

Using $A(10) = 800$ we have

$$800 = 1000e^{50k}$$

so

$$\ln(\frac{4}{5}) = 50k$$

and

$$k = \frac{1}{50}\ln(\frac{4}{5}).$$

Putting this value for k into (*) we have

$$A = 1000\exp(\frac{t^2}{100}\ln(\frac{4}{5})) = 1000\exp\ln((\frac{4}{5})^{t^2/100})$$

so

$$A = 1000(\frac{4}{5})^{t^2/100}.$$

4. A 1000-gallon tank, initially full of water, develops a leak at the bottom. Given that 500 gallons of water leaks out in the first 30 minutes, find the amount of water, $A(t)$ in the tank t minutes after the leak develops if water drains off at a rate proportional to the product of the time elapsed and the square root of the amount of water present.

Solution.

$$A' = kt\sqrt{A}$$
$$A(0) = 1000$$
$$A(30) = 500$$

Solving the differential equation, we have

$$\frac{A'}{\sqrt{A}} = kt$$

$$\int A^{-1/2} dA = \int ktdt + C$$

$$2A^{1/2} = \frac{1}{2}kt^2 + C$$

$A(0) = 1000$ so

$$2 \cdot \sqrt{1000} = 0 + C \text{ so } C = 20\sqrt{10}$$

Continuing,

$$2A^{1/2} = \frac{1}{2}kt^2 + 20\sqrt{10}. \qquad (*)$$

$A(30) = 500$ s0

$$2\sqrt{500} = \frac{1}{2}k \cdot 900 + 20\sqrt{10}$$

$$k = \frac{2(\sqrt{5} - \sqrt{10})}{45}$$

From (*),

$$A^{1/2} = \frac{1}{4}kt^2 + 10\sqrt{10}.$$

So

$$A = \left(\frac{\sqrt{5} - \sqrt{10}}{90} t^2 + 10\sqrt{10} \right)^2 .$$

5. A 1000-gallon tank, initially full of water, develops a leak at the bottom. Given that 300 gallons of water leaks out in the first 20 minutes, find the amount of water, $A(t)$ in the tank t minutes after the leak develops if water drains off at a rate proportional to the square of the amount of water in the tank.

Solution.

$$A' = kA^2$$

$$A(0) = 1000$$

$$A(20) = 700$$

Solving the differential equation we have

$$A^{-2}A' = k$$

$$-A^{-1} = kt + C$$

$$A = \frac{1}{C - kt}.$$

Using $A(0) = 1000$ this becomes

$$1000 = \frac{1}{C} \text{ so } A = \frac{1000}{1 - 1000kt}$$

Using $A(20) = 700$ we have

$$700 = \frac{1000}{1 - 20000k} \text{ so } k = -\frac{3}{140\,000}$$

and

$$A = \frac{1000}{1 - 1000(-\frac{3}{140\,000})t}.$$

So

$$A = \frac{140\,000}{3t + 140}.$$

6. Let $P(t)$ be be the number (in millions) of people who become aware of the product by time t.

$$P' = kt(2 - P)$$

$$P(0) = 0$$

$$P(10) = 0.4$$