Section 3.1

Section 3.1
Introduction to Second Order Linear Differential Equations

Definition. Saying that L is a **second order linear differential operator** over an interval J means that there is a pair of continuous functions (p, q) defined on J such that

$$Ly = y'' + py' + qy$$

whenever y is a twice differential function defined on J. We will be concerned with the **homogeneous** differential equation

$$Ly = 0 \text{ or } y'' + p(x)y' + q(x)y = 0, \qquad 1$$

the **nonhomogeneous** differential equation

$$Ly = f \text{ or } y'' + p(x)y' + q(x)y = f(x), \qquad 2$$

and the initial value problems consisting of (1) or (2) and

$$y(x_0) = k_0 \text{ and } y'(x_0) = k_1$$

where x_0 is a number in J and each of k_0 and k_2 is a number.

The equations (1) and (2) are said to be linear because L has the following properties.

Theorem. $L[cy] = cLy$ and $L[y_1 + y_2] = Ly_1 + Ly_2$.

As a consequence of these properties, we have the following.

Theorem. If L is a second order linear differential operator over an interval J, each of y_1 and y_2 is a twice differentiable function with domain J, and each of c_1 and c_2 is a number, then

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2.$$

Corollary. If L is a second order linear differential operator over an interval J , m is a positive integer, each of y_1, y_2, ... , y_m is a twice differentiable function with domain J, and each of c_1, c_2, ... , c_m is a number, then

$$L(c_1y_1 + c_2y_2 \cdots + c_ny_n) = c_1Ly_1 + c_2Ly_2 + \cdots + c_nLy_n.$$

We will accept the following uniqueness and existence theorem and use it as a basis for developing a description of all solutions to the homogeneous equation in the next section. An indication of proof will be given in a later chapter

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Theorem. Suppose that L is a second order linear differential operator over the interval J. If $x_0$is a number in J and each of k_0 and k_1 is a number, there is a unique function y defined on J such that

$$
\begin{aligned}
Ly &= 0 \text{ on } J, \\
y(x_0) &= k_0, \text{ and} \\
y'(x_0) &= k_1.
\end{aligned}
$$

Theorem. Suppose that L is a second order linear differential operator over the interval J. If $x_0$is a number in J

$$
\begin{aligned}
Ly &= 0 \text{ on } J, \\
y(x_0) &= 0, \text{ and} \\
y'(x_0) &= 0
\end{aligned}
$$

then

$$y(x) = 0 \text{ for all } x \text{ in } J.$$

Proof. The zero function has these properties, and there is only one function with these properties, so y must be the zero function.

Theorem. Suppose that L is a second order linear differential operator that is regular over the interval J. If

$$Lu = 0 \text{ on } J,$$

$$Lv = 0 \text{ on } J,$$

$$u(x_0) = v(x_0), \text{ and}$$

$$u'(x_0) = v'(x_0)$$

for some x_0 in J, then

$$u(x) = v(x) \text{ for all } x \text{ in } J.$$

Proof. There is only one solution to the homogeneous equation satisfying a given list of initial conditions.