Section 3.2
Second Order Linear Homogeneous Differential Equations

In this section L will be a second order linear differential operator over an interval J,

$$Ly = y'' + p(x)y' + q(x)y$$

where each of p and q is continuous on J.

We will be concerned with the homogeneous equation

$$Ly = 0$$

or

$$y'' + p(x)y' + q(x)y = 0.$$

Our main goal is to develop a description of all solutions to this equation.

Theorem. Every linear combination of solutions to the homogeneous equation is also a solution.

If

$$Ly_k = 0 \text{ for } k = 1,2,\ldots,m,$$

each of c_1, c_2, … ,c_m is a number and

$$u = c_1y_{1+}c_2y_2 + \cdots + c_my_m,$$

then

$$Lu = 0.$$

This is true because

$$\begin{aligned} L(c_1y_1 + c_2y_2\cdots + c_ny_n) &= c_1Ly_1 + c_2Ly_2 + \cdots + c_nLy_n \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 \\ &= 0. \end{aligned}$$

Definition. Suppose that each of y_1 and y_2 is a function with domain J. Saying that y_1 and y_2 are **linearly independent** over J means that if each of c_1 and c_2 is a number and

$$c_1y_1(x) + c_2y(x) = 0 \text{ for all } x \text{ in } J$$

then

$$c_1 = c_2 = 0.$$

Saying that y_1 and y_2 are **linearly dependent** means that they are not linearly independent.

Note. y_1 and y_2 are linearly dependent if and only if there is a pair of numbers c_1 and c_2 with at least one of c_1 and c_2 not zero such that

$$c_1 y_1(x) + c_2 y(x) = 0 \text{ for all } x \text{ in } J.$$

Theorem. y_1 and y_2 are linearly dependent if and only if there is a number c such that

$$y_1(x) = c y_2(x) \text{ for all } x \text{ in } J$$

or there is a number d such that

$$y_2(x) = d y_1(x) \text{ for all } x \text{ in } J.$$

Example. Let $y_1(x) = x$ and $y_2(x) = x^2$ for all x. It is the case that y_1 and y_2 are linearly independent.

To verify this, suppose that

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for all } x.$$

Then

$$c_1 x + c_2 x^2 = 0 \text{ for all } x.$$

Letting $x = 1$ then letting $x = -1$ we have

$$c_1 + c_2 = 0 \text{ and } -c_1 + c_2 = 0.$$

From these two equations it follows that $c_1 = 0$ and $c_2 = 0$. So y_1 and y_2 are linearly independent.

Example. Let $y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$. From a trig identity, we know that $\sin 2x = 2 \sin x \cos x$. Thus

$$y_1(x) = 2 y_2(x) \text{ for all } x.$$

Since y_1 is a constant multiple of y_2 it follows that y_1 and y_2 are linearly dependent.

Definition. When (y_1, y_2) is a pair of functions each defined on an interval J, their **Wronski matrix** is given by

$$M_W[y_1, y_2] = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

and their **Wronskian** is given by

$$W[y_1,y_2] = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_2y_1'$$

We need some facts from elementary linear algebra.

Theorem. If each of a, b, c, d, e, and f is a number and

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

there is a unique pair of numbers x and y such that

$$ax + by = e \text{ and}$$
$$cx + dy = f.$$

moreover

$$x = \frac{ed - bf}{ad - bc} \text{ and } y = \frac{af - ec}{ad - bc}.$$

Theorem. If each of a, b, c, and d is a number and

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

the unique pair of numbers x and y such that

$$ax + by = 0 \text{ and}$$
$$cx + dy = 0$$

is (x,y) where

$$x = 0 \text{ and } y = 0.$$

Theorem. If each of a, b, c, and d is a number and

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 0$$

there are numbers x and y at least one of which is **not zero** such that

$$ax + by = 0 \text{ and}$$
$$cx + dy = 0.$$

Theorem. (**First Wronskian Test**) If $W[y_1,y_2](x_0) \neq 0$ for some number x_0 in the interval J where y_1 and y_2 are defined, then y_1 and y_2 are linearly independent over J.

Proof. Suppose that

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for all x in J. Then

$$c_1 y_1'(x) + c_2 y_2'(x) = 0$$

for all x in J. When $x = x_0$ we have

$$y_1(x_0)c_1 + y_2(x_0)c_2 = 0 \text{ and}$$
$$y_1'(x_0)c_1 + y_2'(x_0)c_2 = 0\ .$$

Since

$$W[y_1,y_2](x_0) = \det\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$$

it follows that $c_1 = 0$ and $c_2 = 0$.

Note. Without an additional hypothesis, the first Wronskian test does not detect linear dependence.

Example If $y_1(x) = x^2$ for all x and $y_2(x) = x^2$ for $x \geq 0$ and $y_2(x) = -x^2$ for $x < 0$, then $W[y_1,y_2](x) = 0$ for all x. However, y_1 and y_2 are linearly independent.

Theorem (**Second Wronskin Test**) If $Ly_1 = 0$ and $Ly_2 = 0$ on an interval J and $W[y_1,y_2](x_0) = 0$ for some number x_0 in J, then y_1 and y_2 are linearly dependent over J.

Proof. Let c_1 and c_2 be a pair of numbers at least one of which is not zero such that

$$y_1(x_0)c_1 + y_2(x_0)c_2 = 0 \text{ and} \qquad 1$$
$$y_1'(x_0)c_1 + y_2'(x_0)c_2 = 0\ . \qquad 2$$

This can be done because

$$W[y_1,y_2](x_0) = \det\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 0.$$

Let

$$u(x) = c_1 y(x) + c_2 y_2(x) \text{ for all } x \text{ in } J.$$

Then u is a linear combination of solutions to the homogeneous equation $Ly = 0$, so u is also a solution. From (1) and (2), it follows that

$$u(x_0) = 0 \text{ and } u'(x_0) = 0$$

so

$$u(x) = 0 \text{ for all } x \text{ in } J.$$

We now have

$$c_1 y(x) + c_2 y_2(x) = 0 \text{ for all } x \text{ in } J$$

and at least one of c_1 and c_2 is not zero. This shows that y_1 and y_2 are linearly dependent.

Definition. Saying that (y_1,y_2) or $\{y_1,y_2\}$ is a **fundamental pair** or **fundamental set** for L or for $Ly = 0$ or for

$$y'' + py' + qy = 0$$

on J means that

$$Ly_1 = 0 \text{ or } y_1'' + py_1' + qy_1 = 0,$$

$$Ly_2 = 0 \text{ or } y_2'' + py_2' + qy_2 = 0,$$

and

$$y_1 \text{ and } y_2 \text{ are linearly independent.}$$

Theorem. If $\{y_1,y_2\}$ is a fundamental set or fundamental pair for $Ly = 0$ on J, then

$$Ly = 0 \text{ or } y'' + py' + qy = 0 \text{ on } J$$

if and only if

$$y = c_1 y_1 + c_2 y_2$$

for some pair of numbers $c_1$and c_2.

Proof. If $y = c_1 y_1 + c_2 y_2$ then $Ly = 0$ because every linear combination of solutions to the homogeneous equation is a solution.

If $Ly = 0$, let x_0 be a number in J and let c_1 and c_2 be numbers such that

$$y_1(x_0)c_1 + y_2(x_0)c_2 = y(x_0) \text{ and} \qquad 1$$

$$y_1'(x_0)c_1 + y_2'(x_0)c_2 = y'(x_0)\ . \qquad 2$$

This can be done because by the second Wronskian test, $W[y_1,y_2](x_0) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$; otherwise y_1 and y_2 would be linearly dependent. Let

$$u(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for all } x \text{ in } J.$$

From (1) and (2) it follows that $u(x_0) = y(x_0)$ and $u'(x_0) = y'(x_0)$. Since $Ly = 0$ and $Lu = 0$ it follows that

$$y(x) = u(x) \text{ for all } x \text{ in } J.$$

Thus

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for all } x \text{ in } J.$$

Example. Let

$$y_1(x) = e^{-2x} \text{ and } y_2(x) = e^{3x}$$

for all x. It is easy to verify that each of y_1 and y_2 is a solution to

$$y'' - y' - 6y = 0.$$

$$\begin{aligned} W[y_1, y_2](x) &= y_1(x) y_2'(x) - y_2(x) y_1'(x) \\ &= e^{-2x}(3)e^{3x} - e^{3x}(-2)e^{-2x} \\ &= 5e^{x}. \end{aligned}$$

Thus $W[y_1, y_2](0) = 5e^0 = 5 \neq 0$ implying that y_1 and y_2 are linearly independent. Thus y is a solution to the differential equation if and only if

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}$$

for some pair of numbers c_1 and c_2 and all x.

A Formula for the Wronskian. If $Ly_1 = 0$ and $Ly_2 = 0$ on an interval J,

$$Ly = y'' + p(x)y' + q(x)y,$$

$$w = W[y_1, y_2],$$

and x_0 is a number in J then

$$w' + p(x)w = 0$$

and

$$w(x) = w(x_0) e^{-\int_{x_0}^{x} p(t)dt}$$

for all t in J.

A Formula for a Second Solution. Let L be as above. If $Ly_1 = 0$ on an interval J and $y_1(x) \neq 0$ for all x in J, a second solution y_2 to $Ly = 0$ on J such that (y_1, y_2) is a fundamental pair is given by

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1(x)^2} dx.$$

Leave off the $+C$ in each integration. The formula in Problem 15 for Section 3.2 is not correct. Both anti-derivatives are with respect to x.

Example. It is each to verify that y_1 given by $y_1(x) = e^x$ is a solution to

$$y'' - 2y' + y = 0.$$

A second solution y_2 such that (y_1, y_2) is a fundamental pair is given by

$$\begin{aligned} y_2(x) &= e^x \int \frac{e^{-\int(-2)dx}}{(e^x)^2} dx \\ &= e^x \int \frac{e^{2x}}{e^{2x}} dx \\ &= e^x \int 1 dx \\ &= xe^x. \end{aligned}$$

Additional Examples: See Section 3.2 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.2. The answers are posted on Dr. Walker's web site.