Section 3.3

Section 3.3
Constant Coefficient Homogeneous
Second Order Linear Differential Equations

Definition. Suppose that L is a constant coefficient second order linear differential operator with

$$Ly = y'' + ay' + by$$

where each of a, and b is a real number. The polynomial associated with L is the function $\mathcal{P}$ given by

$$\mathcal{P}(r) = r^2 + ar + b$$

for all complex numbers r. In the text, it is called the **characteristic polynomial.**

$$Ly = y'' + ay' + by$$

$$\mathcal{P}(r) = r^2 + ar + b$$

Theorem. If r is a zero of $\mathcal{P}$ (meaning that $\mathcal{P}(r) = 0$) and

$$y = e^{rx}$$

then

$$Ly = 0$$

i.e.

$$y'' + ay' + by = 0$$

Proof.

$$y = e^{rx} \text{ so } y' = re^{rx} \text{ and } y'' = r^2e^{rx}$$

Thus

$$\begin{aligned} Ly &= r^2e^{rx} + are^{rx} + be^{rx} = (r^2 + ar + b)e^{rx} = \mathcal{P}(r)e^{rx} \\ &= 0e^{rx} = 0 \end{aligned}$$

Theorem. (Case I.) Let L and $\mathcal{P}$ be as above

$$Ly = y'' + ay' + by \text{ and } \mathcal{P}(r) = r^2 + ar + b.$$

If $\mathcal{P}$ has two zeros r_1 and r_2 ($r_1 \neq r_2$), a fundamental set or pair for L is

$$\{e^{r_1 x}, e^{r_2 x}\}$$

and $Ly = 0$ or

$$y'' + ay' + by = 0$$

if and only if

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

for some pair of numbers c_1 and c_2.

Theorem. (Case 2.) Let L and $\mathcal{P}$ be as above

$$Ly = y'' + ay' + by \text{ and } \mathcal{P}(r) = r^2 + ar + b.$$

If $\mathcal{P}$ has only one zero r_0, a fundamental set or pair for L is

$$\{e^{r_0 x}, xe^{r_0 x}\}$$

and $Ly = 0$ or

$$y'' + ay' + by = 0$$

if and only if

$$y = c_1 e^{r_0 x} + c_2 x e^{r_0 x}$$

for some pair of numbers c_1 and c_2.

Theorem. (Case 3.) Let L and $\mathcal{P}$ be as above

$$Ly = y'' + ay' + by \text{ and } \mathcal{P}(r) = r^2 + ar + b$$

If $\mathcal{P}$ has complex zeros $\alpha + \beta i$ and $\alpha - \beta i$ where each of α and β is real and $\beta \neq 0$, a fundamental set or pair for L is

$$\{e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x\}$$

and $Ly = 0$ or

$$y'' + ay' + by = 0$$

if and only if

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

for some pair of numbers c_1 and c_2.

Note.

$$\mathcal{P}(r) = r^2 + ar + b$$

The polynomial $\mathcal{P}$ has two zeros r_1 and r_2 if and only if

$$\mathcal{P}(r) = (r - r_1)(r - r_2)$$

with $r_1 \neq r_2$. This happens if and only if

$$a^2 - 4b \neq 0$$

in which case

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \text{ and } r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Note.

$$\mathcal{P}(r) = r^2 + ar + b$$

The polynomial $\mathcal{P}$ has only one zero r if and only if

$$\mathcal{P}(r) = (r - r_0)^2.$$

This happens if and only if

$$a^2 - 4b = 0$$

in which case

$$r_0 = \frac{-a}{2}.$$

Note.

$$\mathcal{P}(r) = r^2 + ar + b$$

The polynomial $\mathcal{P}$ has complex zeros $\alpha + \beta i$ and $\alpha - \beta i$ where each of α and β is real and $\beta \neq 0$ if and only if

$$a^2 - 4b < 0$$

in which case

$$\alpha = \frac{-a}{2} \text{ and } \beta = \frac{\sqrt{4b - a^2}}{2}.$$

Example. The polynomial $\mathcal{P}$ for

$$y'' + 5y' + 6y = 0$$

is given by

$$\mathcal{P}(r) = r^2 + 5r + 6.$$

$$\mathcal{P}(r) = (r + 2)(r + 3)$$

so the zeros of $\mathcal{P}$ are -2 and -3. A fundamental pair for the d.e. is

$$\{e^{-2x}, e^{-3x}\}$$

so y is a solution to the d.e. if and only if

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

Additional Examples: See Section 3.3 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.3. The answers are posted on Dr. Walker's web site.