Section 3.4

Section 3.4
Nonhomogeneous Second Order Linear Differential Equations
Part 1

In this section and the next, we will be concerned with finding the solutions to the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{N}$$

on an interval J when each of p, q, and f is a continuous function with domain J. In order to solve (N) we will first need to solve the related homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{H}$$

which is sometimes called the **reduced equation**. In connection with (N) and (H) we define the linear differential operator L by

$$Ly = y'' + p(x)y' + q(x)y$$

whenever y is a twice differentiable function with domain J.

Recall that

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2.$$

Consequently,

$$L(y_1 + y_2) = Ly_1 + Ly_2,\ L(y_1 - y_2) = Ly_1 - Ly_2, \text{ and } L(cy) = cLy.$$

Also, if

$$Ly_1 = f \text{ and } Ly_2 = f \text{ then } L(y_1 - y_2) = f - f = 0.$$

The difference of two solutions to (N) is a solution to (H).

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{N}$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{H}$$

In order to find all solutions to the nonhomogeneous equation (N) we need one solution of (N) (called a **particular solution**) and all solutions of the corresponding homogeneous or reduced equation (H).

Theorem. Suppose that

$$Lz = f \text{ or } z'' + p(x)z' + q(x)z = f(x) \text{ on } J.$$

(The function z is called a **particular solution** to the nonhomogeneous equation (N), and the following is a description of all solutions to (N).) It follows that

$Ly = f$ which means $z'' + p(x)z' + q(x)z = f(x)$ on J if and only if

$y = u + z$ for some u such that

$Lu = 0$ which means $u'' + p(x)u' + q(x)u = 0$.

Proof. If $Ly = f$, let $u = y - z$. Then $y = u + z$ and $Lu = L(y - z) = Ly - Lz = f - f = 0$ on J.

If $y = u + z$ and $Lu = 0$ on J, then $Ly = L(u + z) = Lu + Lz = 0 + f = f$ on J.

If $\{y_1, y_2\}$ is a fundamental pair or set for (H), the u in the last theorem can be replaced with

$$c_1 y_1 + c_2 y_2.$$

Theorem. Suppose that $\{y_1, y_2\}$ is a fundamental pair or set for L, and

$$Lz = f \text{ on } J.$$

It follows that

$Ly = f$ on J if and only if

$y = c_1 y_1 + c_2 y_2 + z$ for some pair of numbers c_1 and c_2.

While the solutions to (H) are given by

$$y = c_1 y_1 + c_2 y_2$$

when $\{y_1, y_2\}$ is a fundamental pair, we will see that a particular solution to (N) is of the form

$$z = u \cdot y_1 + v \cdot y_2$$

where each of u and v is a function. Hence the name, Variation of Parameters.

The following theorem gives a formula for a particular solution to the nonhomogeneous equation.

Theorem. Suppose that $\{y_1, y_2\}$ is a fundamental set for(H). Let W be the Wronskian of (y_1, y_2) and let

$$z(x) = y_1(x) \int \frac{-y_2(x)f(x)}{W(x)}dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)}dx \qquad 1$$

or

$$z(x) = y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx \qquad 2$$

It follows that z is a particular solution to (N).

$$z'' + p(x)z' + q(x)z = f(x) \text{ on } J..$$

Note. Leave of the "$+C$" when finding the anti-derivatives. The form (1) is the way the formula is given in the text. See page 94. The equivalent form (2) is easier to remember and we will use it.

Proof.

$$\begin{aligned}
z &= y_2\int \frac{y_1f}{W}dx - y_1\int \frac{y_2f}{W}dx \\
z' &= y_2'\int \frac{y_1f}{W}dx + y_2\frac{y_1f}{W} - y_1'\int \frac{y_2f}{W}dx - y_1\frac{y_2f}{W} \\
z' &= y_2'\int \frac{y_1f}{W}dx - y_1'\int \frac{y_2f}{W}dx \\
z'' &= y_2''\int \frac{y_1f}{W}dx + y_2'\frac{y_1f}{W} - y_1''\int \frac{y_2f}{W}dx - y_1'\frac{y_2f}{W} \\
z'' &= y_2''\int \frac{y_1f}{W}dx - y_1''\int \frac{y_2f}{W}dx + \frac{(y_1y_2' - y_2y_1')f}{W} \\
z'' &= y_2''\int \frac{y_1f}{W}dx - y_1''\int \frac{y_2f}{W}dx + f
\end{aligned}$$

$$\begin{aligned}
z &= y_2\int \frac{y_1f}{W}dx - y_1\int \frac{y_2f}{W}dx \\
z' &= y_2'\int \frac{y_1f}{W}dx - y_1'\int \frac{y_2f}{W}dx \\
z'' &= y_2''\int \frac{y_1f}{W}dx - y_1''\int \frac{y_2f}{W}dx + f
\end{aligned}$$

$$\begin{aligned}
Lz &= z'' + pz' + qz \\
&= (y_2''\int \frac{y_1f}{W}dx - y_1''\int \frac{y_2f}{W}dx + f) \\
&\quad + p(y_2'\int \frac{y_1f}{W}dx - y_1'\int \frac{y_2f}{W}dx) \\
&\quad + q(y_2\int \frac{y_1f}{W}dx - y_1\int \frac{y_2f}{W}dx)
\end{aligned}$$

$$
\begin{aligned}
Lz &= z'' + pz' + qz \\
&= (y_2'' \int \frac{y_1 f}{W} dx - y_1'' \int \frac{y_2 f}{W} dx + f) \\
&\quad + p(y_2' \int \frac{y_1 f}{W} dx - y_1' \int \frac{y_2 f}{W} dx) \\
&\quad + q(y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx)
\end{aligned}
$$

$$
\begin{aligned}
Lz &= (y_2'' + py_2' + qy_2) \int \frac{y_1 f}{W} dx \\
&\quad - (y_1'' + py_1' + qy_1) \int \frac{y_2 f}{W} dx \\
&\quad + f
\end{aligned}
$$

$$
\begin{aligned}
Lz &= (y_2'' + py_2' + qy_2) \int \frac{y_1 f}{W} dx \\
&\quad - (y_1'' + py_1' + qy_1) \int \frac{y_2 f}{W} dx \\
&\quad + f
\end{aligned}
$$

$$Lz = (Ly_2) \int \frac{y_1 f}{W} dx - (Ly_1) \int \frac{y_2 f}{W} dx + f$$

$$Lz = 0 \cdot \int \frac{y_1 f}{W} dx - 0 \cdot \int \frac{y_2 f}{aw} dx + f$$

$$Lz = f$$

Note. If you want to derive the formula, start with

$$z = u\mathrm{y}_1 + vy_2.$$

Assume that

$$z' = u\mathrm{y}_1' + vy_2'.$$

This is equivalent to assuming that

$$y_1 u' + y_2 v' = 0. \qquad 1$$

$$z'' = u\mathrm{y}_1'' + u'y_1' + vy_2'' + v'y_2'.$$

Using these expressions for z, z', z'' together with the fact that $Ly_1 = 0$ and $Ly_2 = 0$, we get $Lz = y_1'u' + y_2'v'$. So since we want $Lz = f$, we have

$$y_1'u' + y_2'v' = f \qquad 2$$

Solve

$$y_1 u' + y_2 v' = 0. \qquad 1$$

and

$$y_1' u' + y_2' v' = f \qquad 2$$

for u' and v'. Multiply (1) by $y_2'$and (2) by y_2. Subtract to get

$$u' = \frac{-y_2 f}{W}.$$

Integrate to get

$$u = \int \frac{-y_2 f}{W} dx.$$

Solve

$$y_1 u' + y_2 v' = 0. \qquad 1$$

and

$$y_1' u' + y_2' v' = f \qquad 2$$

for u' and v'. Multiply (1) by $y_1'$and (2) by y_1. Subtract to get

$$v' = \frac{y_1 f}{W}.$$

Integrate to get

$$v = \int \frac{y_1 f}{W} dx.$$

Since

$$z = u\mathrm{y}_1 + v y_2 = y_1 u + y_2 v$$

we have

$$z(x) = y_1(x) \int \frac{-y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx$$

or

$$z(x) = y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx - y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx.$$

Example. Find all functions y such that

$$y'' + y = \tan x \text{ for } 0 < x < \frac{\pi}{2}$$

Solution. We are looking for all y such that

$$Ly = f \text{ on } J$$

where $Ly = y'' + y$, $f(x) = \tan x$, and J consists of all x where $0 < x < \frac{\pi}{2}$. The reduced equation is

$$y'' + y = 0$$

and fundamental set for L is $\{y_1, y_2\}$ where $y_1(x) = \cos x$ and $y_2(x) = \sin x$. The Wronskian W is given by

$$W(x) = \det\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x + \sin^2 x = 1.$$

A particular solution z satisfying $Lz = f$ is given by

$$\begin{aligned}
z(x) &= y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx \\
&= \sin x \int \frac{\cos x \tan x}{1}dx - \cos x \int \frac{\sin x \tan x}{1}dx \\
&= \sin x \int \sin x dx - \cos x \int \frac{\sin^2 x}{\cos x}dx \\
&= \sin x \int \sin x dx - \cos x \int \frac{1-\cos^2 x}{\cos x}dx \\
&= \sin x \int \sin x dx - \cos x \int (\sec x - \cos x)dx \\
&= \sin x(-\cos x) - \cos x(\ln(\sec x + \tan x) - \sin x) \\
&= -\cos x \ln(\sec x + \tan x).
\end{aligned}$$

$$z(x) = -\cos x \ln(\sec x + \tan x).$$

Thus y is a solution to the given differential equation and only if

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x).$$

for some pair of numbers (c_1, c_2) and all x with $0 < x < \frac{\pi}{2}$.

Note. Sometimes when applying this method, the solution z will turn out to be of the form

$$z = z_1 + z_2$$

where

$$Lz_2 = 0 \text{ on } J.$$

In this case discard $z_2$and use z_1 as the particular solution.

This works because

$$f = Lz = L(z_1 + z_2) = Lz_1 + Lz_2 = 0 + Lz_1 = Lz_1.$$

Example. Find all functions y such that

$$y'' - 3y' + 2y = \frac{1}{1+e^{-x}} \text{ for all } x \text{ in } \mathbb{R}.$$

Solution The reduced equation is

$$y'' - 3y' + 2y = 0.$$

The characteristic polynomial $\mathcal{P}$ is given by

$$\mathcal{P}(r) = r^2 - 3r + 2 = (r-1)(r-2)$$

so a fundamental pair is $\{y_1, y_2\}$ where $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. The Wronskian W is given by

$$W(x) = \det\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} = e^{3x}.$$

A particular solution z satisfying the given nonhomogeneous equation is given by

$$\begin{aligned} z(x) &= y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx \\ &= e^{2x}\int \frac{e^x\frac{1}{1+e^{-x}}}{e^{3x}}dx - e^x\int \frac{e^{2x}\frac{1}{1+e^{-x}}}{e^{3x}}dx \\ &= e^{2x}\int \frac{e^{-x}}{1+e^{-x}}e^{-x}dx - e^x\int \frac{e^{-x}}{1+e^{-x}}dx. \end{aligned}$$

Since

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}$$

the first integrand can be re-written so that

$$z(x) = e^{2x}\int\left(1 - \frac{1}{1+e^{-x}}\right)e^{-x}dx - e^x\int \frac{e^{-x}}{1+e^{-x}}dx$$

$$\begin{aligned} &= -e^x + e^{2x}\ln(1+e^{-x}) + e^x\ln(1+e^{-x}) \\ &= e^{2x}\ln(1+e^{-x}) + e^x\ln(1+e^{-x}) - e^x. \end{aligned}$$

Note that

$$z = z_1 + z_2$$

where

$$z_1(x) = e^{2x}\ln(1+e^{-x}) + e^x\ln(1+e^{-x})$$

and

$$z_2(x) = -e^{x.}$$

Since z_2 is a linear combination of y_1 and y_2 ($z_2 = (-1)y_1 + 0\cdot y_2$),

$$Lz_2 = 0,$$

and consequently,

$$Lz_1 = f.$$

Thus

$$Ly = f \text{ on } \mathbb{R}$$

if and only if

$$y(x) = c_1e^x + c_2e^{2x} + e^{2x}\ln(1 + e^{-x}) + e^x\ln(1 + e^{-x})$$

for some pair of numbers (c_1, c_2) and all real numbers x.

Additional Examples: See Section 3.4 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.4. The answers are posted on Dr. Walker's web site.