Section 3.4
Nonhomogeneous Second Order Linear Differential Equations
Part 1

In this section and the next, we will be concerned with finding the solutions to the nonhomogeneous equation

N

on an interval J when each of p, q, and f is a continuous function with domain J. In order to solve (N) we will first need to solve the

H

which is sometimes called the . In connection with (N) and (H) we define the linear differential operator L by

whenever y is a twice differentiable function with domain J.

Recall that

$$L(c_1y_1 + c_2y_2) =$$

Consequently,

, and .

Also, if

and then .

The

N

H

In order to find all solutions to the nonhomogeneous equation (N) we need (called a) and or).

Theorem. Suppose that

(The function z is called a to the nonhomogeneous equation (N), and the following is a description of (N).) It follows that

if and only if

for some u such that

which means .

Proof. If $Ly = f$, let . Then and = = on J.

If and on J, then = = .

If $\{y_1, y_2\}$ is a fundamental pair or set for (H), the u in the last theorem can be replaced with

Theorem. Suppose that $\{y_1, y_2\}$ is a fundamental pair or set for L, and

$$Lz = f \text{ on } J.$$

It follows that

if and only if

for

While the solutions to (H) are given by

when $\{y_1, y_2\}$ is a fundamental pair, we will see that a particular solution to (N) is of the form

where each of u and v is a function. Hence the name,

The following theorem gives a formula for a particular solution to the nonhomogeneous equation.

Theorem. Suppose that $\{y_1, y_2\}$ is a fundamental set for(H). Let W be the Wronskian of (y_1, y_2) and let

$$z(x) = \qquad (1)$$

or

$z(x) =$

It follows that z is a particular solution to (N).

.

Note. Leave of the "$+C$" when finding the anti-derivatives. The formula is given in the text. . The equivalent form (2) is easier to remember and .

Proof.

$$z =$$

$$z' =$$

$$z' =$$

$$z'' =$$

$$z'' =$$

$$z'' =$$

$$z =$$

$$z' =$$

$$z'' =$$

$$Lz = z'' + pz' + qz$$

$$=$$

$$+$$

$$+$$

$Lz =$

$Lz =$

$Lz =$

$Lz =$

Note. If you want to derive the formula, start with

Assume that

This is equivalent to assuming that

. 1

.

Using these expressions for z, z', z'' together with the fact that $Ly_1 = 0$ and $Ly_2 = 0$, we get . So since we want $Lz = f$, we have

2

Solve

1

and

2

for u' and v'. Multiply (1) by y_2' and (2) by y_2. Subtract to get

.

Integrate to get

.

Solve

1

and

2

for u' and v'. Multiply (1) by y_1' and (2) by y_1. Subtract to get

.

Integrate to get

Since

we have

or

Example. Find all functions y such that

Solution. We are looking for all y such that

$$Ly = f \text{ on } J$$

where $Ly =$, , and . The reduced equation is and fundamental set for L is $\{y_1, y_2\}$ where and . The Wronskian W is given by

$$W(x) = \quad = \quad .$$

A particular solution z satisfying $Lz = f$ is given by

$$z(x) =$$
$$=$$
$$=$$
$$=$$
$$=$$
$$=$$
$$=$$

.

Thus y is a solution to the given differential equation and only if

$$y(x) = \quad .$$

for some pair of numbers (c_1, c_2) and all x with $0 < x < \frac{\pi}{2}$.

Note. Sometimes when applying this method, the solution z will turn out to be of the form

where

.

In this case as the particular solution.

This works because

$$f = Lz = \quad = \quad = \quad = \quad .$$

Example. Find all functions y such that for all x in $\mathbb{R}$.

Solution The reduced equation is

$$= 0.$$

The characteristic polynomial $\mathcal{P}$ is given by

$$=$$

so a fundamental pair is $\{y_1, y_2\}$ where and . The Wronskian W is given by

$$W(x) = \quad = \quad .$$

A particular solution z satisfying the given nonhomogeneous equation is given by

$$z(x) =$$

$$=$$

$$= \quad .$$

Since

the first integrand can be re-written so that

$$=$$

$$= \quad .$$

Note that

where

and

Since z_2 is a linear combination of y_1 and y_2 , and consequently,

Thus

on $\mathbb{R}$

if and only if

$$y(x) =$$

for some pair of numbers (c_1, c_2) and all real numbers x.

Additional Examples:

Suggested Problems.