Section 3.5
The Method of Undetermined Coefficients

Note. In this section, L will be a constant coefficient second order linear differential operator with

$$Ly = y'' + ay' + by$$

where each of a, and b is a real number. P will be the polynomial associated with or characteristic polynomial for L.

$$P(r) = r^2 + ar + b.$$

When L is a second order **constant coefficient** and f is of **certain types**, the form of a particular z solution to the nonhomogeneous equation

$$Ly = f$$

can be predicted. This form will contain coefficients that must be determined in order to specify the solution.

$$P(r) = r^2 + ar + b$$

Definition. Saying that r_0 is a zero of P means that

$$P(r_0) = 0.$$

Saying that $r_0$is a zero of **multiplicity one** for P means that

$$P(r) = (r - r_0)(r - r_1) = (r - r_0)^1(r - r_1)^1 \text{ where } r_1 \neq r_0.$$

Saying that $r_0$is a zero of **multiplicity two** for P means that

$$P(r) = (r - r_0)^2$$

Case 1. If

$$f(x) = ke^{r_0 x}$$

where k is a constant and r_0 is not a zero of P (This is equivalent to saying that $e^{r_0 x}$is not a solution to the related homogeneous or reduced equation $Ly = 0.$), there will be a particular solution z satisfying

$$Lz = f$$

of the form

$$z = Ae^{r_0 x}.$$

The value of the coefficient A is found by putting this form into the equation $Lz = f$ and after simplification, equating unknown coefficients and known coefficients on like functions.

Example. Find all solutions to

$$y'' - y' - 6y = 4e^{5x}.$$

Solution. Here $P(r) = r^2 - r - 6 = (r+2)(r-3)$ so the zeros of P are -2 and 3. The number 5 is not a zero of P. e^{5x} is not a solution to $Ly = 0$. There will be a particular solution z of the form Ae^{5x}.

$$z = Ae^{5x}$$
$$z' = 5Ae^{5x}$$
$$z'' = 25Ae^{5x}$$

$$Lz = z'' - z' - 6z = 25Ae^{5x} - 5Ae^{5x} - 6Ae^{5x}$$
$$= 14Ae^{5x}$$

This will be the right side of the given differential equation if and only if

$$14A = 4 \text{ or } A = \frac{2}{7}$$

Thus there is a particular solution z given by

$$z = \frac{2}{7}e^{5x}.$$

Since the zeros of the associated polynomial are -2 and 3, a fundamental pair for the related homogeneous or reduced equation is e^{-2x} and e^{3x}. Thus

$$y'' - y' - 6y = 4e^{5x}.$$

if and only if

$$y = c_1e^{-2x} + c_2e^{3x} + \frac{2}{7}e^{5x}.$$

Case 2. If

$$f(x) = q(x)e^{r_0x}$$

where q is a polynomial of degree n and r_0 is not a zero of P (This is equivalent to saying that e^{r_0x} is not a solution to the related homogeneous or reduced equation $Ly = 0$.), there will be a particular solution z satisfying

$$Lz = f$$

of the form

$$z = Q(x)e^{r_0x}$$

where Q is a polynomial of degree n with coefficients that are to be determined. The values of the coefficients are found by putting this form into the equation $Lz = f$ and after simplification, equating unknown coefficients and known coefficients on like functions. Case 1 is included in Case 2 because q can be a constant polynomial.

Example.
Give the form of a particular solution to

$$y'' - y' - 6y = (5 - 2x^2)e^{5x}.$$

Solution. Here $P(r) = r^2 - r - 6 = (r+2)(r-3)$ so the zeros of P are -2 and 3. The number 5 is not a zero of P. $e^{5x}$is not a solution to $Ly = 0$. The polynomial q is of degree 2. There will be a particular solution z of the form

$$z = (A + Bx + Cx^2)e^{5x}$$

Case 3. If

$$f(x) = q(x)e^{r_0 x}$$

where q is a polynomial of degree n and r_0 is a zero of P, let m be its multiplicity. m is one if $e^{r_0 x}$ is a solution and $xe^{r_0 x}$ is not solution to the related homogeneous or reduced equation. m is two if $xe^{r_0 x} is$a solution to the related homogeneous or reduced equation. There will be a particular solution z satisfying

$$Lz = f$$

of the form

$$z = x^m Q(x)e^{r_0 x}$$

where Q is a polynomial of degree n with coefficients that are to be determined. The values of the coefficients are found by putting this form into the equation $Lz = f$ and after simplification, equating unknown coefficients and known coefficients on like functions. The case where q is a constant is included here.

Case 4. Suppose that each of α and β is a real number with $\beta \neq 0$ and

$$f(x) = q_1(x)e^{\alpha x}\cos\beta x + q_2(x)e^{\alpha x}\sin\beta x$$

where each of q_1 and q_2 is a polynomial of degree at most n or is the zero function. If $\alpha + \beta i$ is not a zero of P let $m = 0$. If $\alpha + \beta i$ is a zero, let m be its multiplicity. (Since L is second order with real coefficients, this will mean $m = 1$. Later when L is of higher order $m > 1$ will be possible.) There will be a particular solution z satisfying

$$Lz = f$$

i.e.

$$y'' + ay' + by = q_1(x)e^{\alpha x}\cos\beta x + q_2(x)e^{\alpha x}\sin\beta x$$

of the form

$$z = x^m Q_1(x) e^{\alpha x}\cos\beta x + x^m Q_2(x) e^{\alpha x}\sin\beta x$$

where each of Q_1 and Q_2 is a polynomial of degree n. The coefficients in Q_1 and Q_2 can be found by inserting the form for z into $Lz = f$ and, after simplification, equating unknown and known coefficients on like functions.

$$z = x^m Q_1(x) e^{\alpha x}\cos\beta x + x^m Q_2(x) e^{\alpha x}\sin\beta x$$

The entire form for z must be used even if f contains only sine terms or only cosine terms.

Example. Find all solutions to

$$y'' - y' - 6y = 3\cos 2x.$$

Solution. $P(r) = r^2 - r - 6 = (r+2)(r-3)$ so the zeros of P are -2 and 3. Think of the right side of the differential equation as

$$f(x) = 3e^{0x}\cos 2x + 0e^{0x}\sin 2x.$$

$0 + 2i$ is not a zero of P , 3 is a constant polynomial (degree 0)and the coefficient on $e^{0x}\sin 2x$ can be regarded as the zero polynomial; so $m = 0$, $Q_1(x) = A$, $Q_2(x) = B$, and $z = Ae^{0x}\cos 2x + Be^{0x}\sin 2x$.

$$\begin{aligned} z &= A\cos 2x + B\sin 2x \\ z' &= -2A\sin 2x + 2B\cos 2x \\ z'' &= -4A\cos 2x - 4B\sin 2x \end{aligned}$$

$$\begin{aligned} Lz &= z'' - z' - 6z \\ &= (-4A - 2B - 6A)\cos 2x + (-4B + 2A - 6B)\sin 2x \\ &= (-10A - 2B)\cos 2x + (-10B + 2A)\sin 2x \end{aligned}$$

This will be $f(x) = 3\cos 2x + 0\sin 2x$ if and only if

$$-10A - 2B = 3 \text{ and}$$
$$-10B + 2A = 0$$

or

$$-10A - 2B = 3 \text{ and}$$
$$10A - 50B = 0$$

or

$$A = -\frac{15}{52} \text{ and } B = -\frac{3}{52}.$$

Thus

$$z = -\frac{15}{52}\cos 2x - \frac{3}{52}\sin 2x$$

and y is a solution to the given differential equation if and only if

$$y(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{15}{52}\cos 2x - \frac{3}{52}\sin 2x.$$

Example. Find all solutions to

$$y'' + 2y' + y = x^2 e^{-x}$$

Solution. $P(r) = r^2 + 2r + 1 = (r+1)^2$ so P has only one zero, -1, and it is of multiplicity 2. The right side of the differential equation is

$$f(x) = (0 + 0x + x^2)e^{-x}.$$

Thus a particular solution z will be of the form

$$z = x^2(A + Bx + Cx^2)e^{-x}$$

so

$$z = (Ax^2 + Bx^3 + Cx^4)e^{-x}.$$

Computation shows that

$$z' = (2Ax + (3B - A)x^2 + (4C - B)x^3 - Cx^4)e^{-x}$$

$$z'' = (2A + (6B - 4A)x + (A + 12C)x^2 + (B - 8C)x^3 + Cx^4)e^{-x}$$

and

$$Lx = z'' + 2z' + z = (2A + 6Bx + 12Cx^2)e^{-x}.$$

This will be the given

$$f(x) = (0 + 0x + x^2)e^{-x}$$

if and only if $2A = 0$, $6B = 0$, and $12C = 1$ if and only if $A = 0$, $B = 0$, and $C = \frac{1}{12}$. Thus

$$z = \frac{1}{12}x^4 e^{-x}$$

and y is a solution to the given differential equation if and only if

$$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{12}x^4 e^{-x}.$$

Note. If the method of undetermined coefficients can be used to solve

$$Ly = f_k$$

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for $k = 1, \ldots, n$ then it can be used to solve

$$Ly = f_1 + \cdots + f_n$$

and a particular solution will be of a form that is a sum of the forms for (#). Use all different letters in the predicted form.

Example. Give the form of a particular solution to

$$y'' - 9y' + 20y = -2\cos 2x + 4e^{5x} - 2$$

Solution.

$$z = A\cos 2x + B\sin 2x + Cxe^{5x} + D$$

Note that the 2 on the right side of the given equation can be thought of as $2e^{0x}$so the corresponding part of the form for z is $De^{0x} = D$.

Note. If L is not constant coefficient or if f is not a linear combination of the types considered above, the integral formula method of variation of parameters must be used to solve

$$Ly = f.$$

Example.The method of undetermined coefficients cannot be used to solve

$$y'' - 10y' + 25y = \frac{e^{5x}}{x^2}$$

Additional Examples: See Section 3.5 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.5. The answers are posted on Dr. Walker’s web site.