Section 3.6
Vibrating Mechanical Systems

Suppose that a weight of mass m is suspended by a spring with spring constant k, and the weight moves only in the vertical direction. Let $u(t)$ be the displacement of the weight from the suspension point at time t. Note that $u'(t)$ is the velocity and $u''(t)$ is the acceleration of the weight at time t. Take the downward direction to be positive, and let the magnitude of acceleration due to gravity be g. Let u_0 be the position of the weight when the spring is unstretched and uncompressed. Let u_1 be the position of the weight when the system is in equilibrium. Using Newton's second law and Hooke's law we have

$$mu'' = mg - k(u - u_0)$$

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Let y be the displacement of the weight from the equilibrium position.

$$y = u - u_1$$

Note that $y' = u'$ and $y'' = u''$. Since

$$mg = k(u_1 - u_0)$$

for equilibrium, we have

$$my'' = k(u_1 - u_0) - k(u - u_0) = -k(u - u_1) = -ky.$$

Thus

$$my'' = -ky.$$

If there is damping proportional to velocity, the differential equation becomes

$$my'' = -ky - cy'$$

where c is a positive constant. There might also be an external force applied. For example, the weight might be ferromagnetic and move up and down inside a solenoid to which an alternating EMF is applied. In this case the differential equation becomes

$$my'' = -ky - cy' + G(t).$$

Undamped Free Vibrations

When there is no damping and no applied force the differential equation

$$my'' = -ky$$

will be written

$$y'' + \omega^2 y = 0 \tag{1}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

In this case the weight is said to execute **simple harmonic motion**.

The polynomial P for

$$y'' + \omega^2 y = 0 \tag{1}$$

is given by

$$P(r) = r^2 + \omega^2$$

Its zeros are ωi and $-\omega i$ so y is a solution to (1) if and only if

$$y = c_1 \cos \omega t + c_2 \sin \omega t \tag{2}$$

for some pair of numbers c_1 and c_2.

$$y'' + \omega^2 y = 0 \tag{1}$$

$$y = c_1 \cos \omega t + c_2 \sin \omega t \tag{2}$$

It is also true that y is a solution to (1) if and only if

$$y = A \sin(\omega t + \varphi_0) \tag{3}$$

for some number $A \geq 0$ and number φ_0 with $0 \leq \varphi_0 < 2\pi$.

To get the connection between

$$y = c_1 \cos \omega t + c_2 \sin \omega t \tag{2}$$

and

$$y = A \sin(\omega t + \varphi_0) \tag{3}$$

start with (3) and use the formula for the sine of a sum.

$$\begin{aligned} y &= A(\sin \omega t \cos \varphi_0 + \cos \omega t \sin \varphi_0) \\ &= A \sin \varphi_0 \cos \omega t + A \cos \varphi_0 \sin \omega t \\ &= c_1 \cos \omega t + c_2 \sin \omega t \end{aligned}$$

where $c_1 = A \sin \varphi_0$ and $c_2 = A \cos \varphi_0$.

To go from (2) to (3)

$$y = c_1 \cos \omega t + c_2 \sin \omega t \tag{2}$$

$$y = A\sin(\omega t + \varphi_0) \qquad 3$$

let

$$A = \sqrt{c_1^2 + c_2^2}$$

and choose φ_0 so that

$$A\sin\varphi_0 = c_1 \text{ and } A\cos\varphi_0 = c_2.$$

Assuming $c_2 \neq 0$ this means

$$\tan\varphi_0 = \frac{A\sin\varphi_0}{A\cos\varphi_0} = \frac{c_1}{c_2}$$

so

$$\varphi_0 = \text{Arctan}\frac{c_1}{c_2} \text{ or } \varphi_0 = \pi + \text{Arctan}\frac{c_1}{c_2}$$

With

$$y = c_1\cos\omega t + c_2\sin\omega t \qquad 2$$

and

$$y = A\sin(\omega t + \varphi_0) \qquad 3$$

A is called the **amplitude** and φ_0 is called the phase constant. The **period** T is $\frac{2\pi}{\omega}$, the **frequency** f is $\frac{1}{T} = \frac{\omega}{2\pi}$, and the **angular frequency** is ω.

Damped Free Vibrations

When there is damping proportional to velocity and no applied force, the differential equation

$$my'' = -ky - cy'$$

will be written

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad 4$$

The polynomial P for this equation is given by

$$P(r) = r^2 + \frac{c}{m}r + \frac{k}{m}.$$

Its zeros are

$$\frac{-\frac{c}{m} + \sqrt{(\frac{c}{m})^2 - 4\frac{k}{m}}}{2} = \frac{-c + \sqrt{c^2 - 4km}}{2m} \text{ and } \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

When $c^2 - 4km < 0$, we have what is known as the **underdamped** case. The zeros of P are $\alpha + \beta i$ and $\alpha - \beta i$ where

$$\alpha = \frac{-c}{2m} \text{ and } \beta = \frac{\sqrt{4km - c^2}}{2m}$$

so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t \tag{5}$$

It is also true that y a solution to (4) if and only if

$$y = Ae^{\alpha t} \sin(\beta t + \varphi_0) \tag{6}$$

for some number $A > 0$ and number φ_0 with $0 \leq \varphi_0 < 2\pi$. Note that since $c > 0$, it follows that $\alpha < 0$. The weight continues to move up and down through the equilibrium level but with decreasing amplitude.

When $c^2 - 4km = 0$, we have what is known as the **critically underdamped** case. P has only one zero, $r_0 = \frac{-c}{2m}$, so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = c_1 e^{r_0 t} + c_2 t e^{r_0 t} \tag{7}$$

for some pair of numbers $c_1$and c_2. Since $r_0 < 0$ it follows that

$$\lim_{t \to \infty} y(t) = 0.$$

When $c^2 - 4km > 0$, we have what is known as the **overdamped** case. P has two zeros,

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \text{ and } r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m},$$

so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{8}$$

for some pair of numbers c_1 and c_2. Since each of r_1 and r_2 is negative it follows that

$$\lim_{t \to \infty} y(t) = 0.$$

Undamped Forced Vibrations

We consider next the case where there is no damping and a sinusoidal external applied force. We will take

$$G(t) = F_0 \cos \gamma t$$

so that

$$my'' = -ky + G(t)$$

becomes

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t \qquad 5$$

where

$$\omega = \sqrt{\frac{k}{m}}\,.$$

The method of undetermined coefficients can be used to find a particular solution to (5). We found all solutions to the reduced equation when we considered undamped free vibrations.

$\gamma/2\pi$ is called the applied frequency and $\omega/2\pi$ is called the natural frequency. When $\gamma \neq \omega$, y is a solution to

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t \qquad 5$$

if and only if

$$y = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t \qquad 9$$

for some pair of numbers c_1 and c_2. In this case, it is also true that y is a solution to (5) if and only if

$$y = A \sin(\omega t + \varphi_0) + \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t \qquad 10$$

for some $A > 0$ and φ_0 with $0 \leq \varphi_0 < 2\pi$. In this case the oscillations are bounded but large in magnitude if γ is close to ω.

When $\gamma = \omega$, we have a situation known as **resonance**. y is a solution to

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t \qquad 5$$

if and only if

$$y = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0/m}{2\omega} t \sin \gamma t$$

for some pair of numbers c_1 and c_2. In this case, it is also true that y is a solution to (5) if and only if

$$y = A\sin(\omega t + \varphi_0) + \frac{F_0/m}{2\omega} t \sin\gamma t \quad 11$$

for some $A > 0$ and φ_0 with $0 \leq \varphi_0 < 2\pi$. The oscillations increase in magnitude without bound as t increases.

Damped Forced Vibrations

We consider next the case where there is damping and a sinusoidal external applied force so that

$$my'' = -ky - cy' + G(t)$$

becomes

$$y'' + cy' + \omega^2 y = \frac{F_0}{m}\cos\gamma t \quad 12$$

where

$$\omega = \sqrt{\frac{k}{m}}\,.$$

The method of undetermined coefficients can be used to show that a particular solution z to (12) is given by

$$z(t) = \frac{F_0 m(\omega^2 - \gamma^2)}{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}\cos\gamma t + \frac{F_0 c\gamma}{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}\sin\gamma t$$

which can also be expressed by

$$z(t) = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}}\cos(\gamma t + \psi_0).$$

So y is a solution to (12) if and only if

$$y(t) = u(t) + z(t)$$

where u is a solution to the related homogeneous or reduced equation

$$y'' + cy' + \omega^2 y = 0.$$

Resonance also occurs in this case. The amplidude of the steady state solution z given by

$$z(t) = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}}\cos(\gamma t + \psi_0).$$

is at a maximum when $\gamma = \omega$ where it becomes

$$\frac{F_0}{c\omega}.$$

The maximum amplitude varies inversely with the ampunt of damping.

Each such solution u to the related homogeneous or reduced equation is called a **transient solution**, and the particular solution z is called the **steady state** solution.

We have seen in the Damped Free Vibration case that

$$\lim_{t\to\infty} y(t) = 0$$

for each solution to

$$y'' + cy' + \omega^2 y = 0,$$

hence the name transient solution in this case.

Additional Examples: See Section 3.6 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.6. The answers are posted on Dr. Walker's web site.