Section 3.6

Section 3.6
Vibrating Mechanical Systems

Suppose that a weight of mass m is suspended by a spring with spring constant k, and the weight moves only in the vertical direction. . Note that and . Take the and let the magnitude of . when the spring is . Using we have

Let y be the displacement of the weight from the equilibrium position.

Note that and . Since

for equilibrium, we have

$$my'' = \quad - \quad = \quad =$$

Thus

If there is damping proportional to velocity, the differential equation becomes

where c is a positive constant. There might also be an external force applied. For example, . In this case the differential equation becomes .

Undamped Free Vibrations

When there is no damping and no applied force the differential equation

will be written

1

where

In this case the weight is said to execute .

The polynomial P for

$$y'' + \omega^2 y = 0 \quad (1)$$

is given by

Its zeros are so y is a solution to (1) if and only if

2

for some pair of numbers c_1 and c_2.

$$y'' + \omega^2 y = 0 \quad (1)$$

$$y = c_1 \cos \omega t + c_2 \sin \omega t \quad (2)$$

It is also true that y is a solution to (1) if and only if

3

for some number $A \geq 0$ and number φ_0 with $0 \leq \varphi_0 < 2\pi$.

To get the connection between

2

and

3

start with (3) and use the formula for the sine of a sum.

$y =$

$=$

$=$

where $c_1 =$ and $c_2 =$

To go from (2) to (3)

$$y = c_1 \cos \omega t + c_2 \sin \omega t \quad (2)$$

$$y = A\sin(\omega t + \varphi_0) \qquad 3$$

let

and choose φ_0 so that

and .

Assuming $c_2 \neq 0$ this means

$$= \quad =$$

so

$\varphi_0 =$ or $\varphi_0 =$

With

$$y = c_1\cos\omega t + c_2\sin\omega t \qquad 2$$

and

$$y = A\sin(\omega t + \varphi_0) \qquad 3$$

A is called the and φ_0 is called the . The , the , and the

Damped Free Vibrations

When there is damping proportional to velocity and no applied force, the differential equation

$$my'' = -ky - cy'$$

will be written

4

The polynomial P for this equation is given by

$P(r) =$.

Its zeros are

$=$ and .

When $c^2 - 4km < 0$, we have what is known as the . The zeros of P are where and $\beta =$

so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = \tag{5}$$

It is also true that y a solution to (4) if and only if

$$y = \tag{6}$$

for some number $A > 0$ and number φ_0 with $0 \leq \varphi_0 < 2\pi$. Note that since $c > 0$, it follows that $\alpha < 0$. but with .

When $c^2 - 4km = 0$, we have what is known as the case. P has only one zero, , so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = \tag{7}$$

for some pair of numbers $c_1$and c_2. Since $r_0 < 0$ it follows that

$$\lim_{t\to\infty} y(t) = 0.$$

When $c^2 - 4km > 0$, we have what is known as the case. P has two zeros,

$$r_1 = \quad \text{and } r_2 = \quad ,$$

so y is a solution to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \tag{4}$$

if and only if

$$y = \tag{8}$$

for some pair of numbers c_1 and c_2. Since it follows that

$$\lim_{t\to\infty} y(t) = 0.$$

Undamped Forced Vibrations

We consider next the case where there is no damping and a sinusoidal external applied force. We will take

$$G(t) =$$

so that

$$my'' = -ky + G(t)$$

becomes

5

where

.

The method of undetermined coefficients can be used to find a particular solution to (5). We found all solutions to the reduced equation when we considered undamped free vibrations.

$\gamma/2\pi$ is called the and $\omega/2\pi$ is called the . When is a solution to

$$y'' + \omega^2 y = \frac{F_0}{m}\cos\gamma t$$ 5

if and only if

$$y =$$ 9

for some pair of numbers c_1 and c_2. In this case, it is also true that y is a solution to (5) if and only if

$$y =$$ 10

for some $A > 0$ and φ_0 with $0 \leq \varphi_0 < 2\pi$. In this case the oscillations are bounded but large in magnitude if γ is close to ω.

When $\gamma = \omega$, we have a situation known as . y is a solution to

5

if and only if

$$y =$$

for some pair of numbers c_1 and c_2. In this case, it is also true that y is a solution to (5) if and only if

$$y = \qquad (11)$$

for some $A > 0$ and φ_0 with $0 \leq \varphi_0 < 2\pi$. The oscillations increase in magnitude without bound as t increases.

Damped Forced Vibrations

We consider next the case where there is damping and a sinusoidal external applied force so that

$$my'' = -ky - cy' + G(t)$$

becomes

$$\qquad (12)$$

where

The method of undetermined coefficients can be used to show that a particular solution z to (12) is given by

$$z(t) =$$

which can also be expressed by

$$z(t) = \quad .$$

So y is a solution to (12) if and only if

where u is a solution to the related homogeneous or reduced equation

.

Resonance also occurs in this case. The amplidude of the steady state solution z given by

$$z(t) =$$

is at a maximum when $\gamma = \omega$ where it becomes

The maximum amplitude varies inversely with the ampunt of damping.

Each such solution u to the related homogeneous or reduced equation is called a , and the particular solution z is called the .

We have seen in the Damped Free Vibration case that

$$\lim_{t \to \infty} y(t) = 0$$

for each solution to

$$y'' + cy' + \omega^2 y = 0,$$

hence the name transient solution in this case.

Additional Examples: See Section 3.6 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.6. The answers are posted on Dr. Walker's web site.