Section 3.7
Higher Order Linear Differential Equations

Definition. When n is a positive integer, saying that L is an n^{th} **order linear differential operator** over an interval J means that there is a list of continuous functions $(p_0, p_1, \ldots, p_{n-1})$ each defined on J such that

$$Ly = y^{(n)} + p_{n-1}y^{(n-1)} + p_{n-2}y^{(n-2)} + \cdots + p_1 y' + p_0 y$$

whenever y is an n-times differentiable function defined on J. We will be concerned with the **homogeneous** differential equation

$$Ly = 0, \qquad 1$$

the **nonhomogeneous** differential equation

$$Ly = f, \qquad 2$$

and the initial value problems consisting of (1) or (2) and

$$y(x_0) = k_0 \text{ and } y^{(j)}(x_0) = k_j \text{ for } j = 1, 2, \ldots, n-1$$

where x_0 is a number in J and each of $k_0, k_1, \ldots, k_{n-1}$ is a number.

Theorem. If L is an n^{th}order linear differential operator over an interval J,each of y_1 and y_2 is an n-times differentiable function with domain J, and each of c_1 and c_2 is a number, then

$$L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2.$$

Special cases are

$$L(y_1 + y_2) = L(y_1) + L(y_2) \text{ and } L(cy) = cL(y).$$

Corollary. If L is an n^{th} order linear differential operator over an interval J , m is a positive integer, each of y_1, y_2, ... , y_m is an n-times differentiable function with domain J, and each of c_1, c_2, ... , c_m is a number, then

$$L(c_1 y_1 + c_2 L y_2 \cdots + c_m y_m) = c_1 L y_1 + c_2 L y_2 + \cdots + c_m L y_m.$$

We will accept the following uniqueness and existence theorem and use it as a basis for developing a description of all solutions to the homogeneous equation. An indication of proof will be given in a later chapter

.

Theorem. Suppose that L is an n^{th} order linear differential operator over the interval J. If x_0 is a number in J and each of $k_0, k_1, \ldots, k_{n-1}$ is a number, there is a unique function y defined on J such that

$$Ly = 0 \text{ on } J, \text{ and}$$
$$y^{(j)}(x_0) = k_j \text{ for } j = 0, 1, \ldots, n-1.$$

Theorem. Every linear combination of solutions to the homogeneous equation is also a solution.

If

$$Ly_k = 0 \text{ for } k = 1, 2, \ldots, m,$$

each of $c_1, c_2, \ldots, c_m$ is a number and

$$u = c_1y_{1+}c_2y_2 + \cdots + c_my_m,$$

then

$$Lu = 0.$$

This is true because

$$\begin{aligned} L(c_1y_1 + c_2y_2 \cdots + c_ny_n) &= c_1Ly_1 + c_2Ly_2 + \cdots + c_nLy_n \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 \\ &= 0. \end{aligned}$$

Homogeneous Equations

Definition. Suppose that y_k is a function with domain J for $k = 1, \ldots, m$. Saying that $(y_1, \ldots, y_m)$ is a list of functions that are **linearly independent** over J means that if each of $c_1, \ldots c_m$ is a number and

$$c_1y_1(x) + \cdots + c_my_m(x) = 0 \text{ for all } x \text{ in } J$$

then

$$c_1 = c_2 = \cdots = c_m = 0.$$

Saying that $(y_1, \ldots, y_m)$ is a list of functions that are **linearly dependent** over J means that it is a list of functions that are **not** linearly independent.

Note. $(y_1, \ldots, y_m)$ is a list of functions that are **linearly dependent** over J means if and only if there is a list of numbers $c_1, \dots, c_m$ at least one of which is not zero such that

$$c_1 y_1(x) + \cdots + c_m y_m(x) = 0 \text{ for all } x \text{ in } J.$$

Definition. When $(y_1, \dots, y_n)$ is a of functions each defined on an interval J and each having $n - 1$ derivatives, their **Wronski matrix** is given by

$$M_W[y_1, \dots, y_n] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

and their **Wronskian** is given by

$$W[y_1, \dots, y_n] = \det\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Theorem. (**First Wronskian Test**) If $W[y_1, \dots, y_n](x_0) \neq 0$ for some number x_0 in the interval J, then $(y_1, \dots, y_n)$ a list of functions that are linearly independent over J.

Theorem (**Second Wronskin Test**) If $Ly_k = 0$ on an interval J for $k = 1, \dots n$ and $W[y_1, \dots, y_n](x_0) = 0$ for some number x_0 in J, then $(y_1, \dots, y_n)$ a list of functions that are linearly dependent over J.

Definition. Saying that $(y_1, \dots, y_n)$ or $\{y_1, \dots, y_n\}$ is a **fundamental list** or **fundamental set** for L or for $Ly = 0$ means that

$$Ly_k = 0 \text{ for } k = 1, \dots n$$

and

$$(y_1, \dots, y_n) \text{ is a list of functions that are linearly independent over } J.$$

The following theorem gives a description of all solutions to the homogeneous equation.

Theorem. If $(y_1,\dots,y_n)$ is a fundamental list for L then
$$Ly = 0$$
if and only if
$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
for some list of numbers
$$(c_1,c_2,\dots,c_n)$$

Definition. When L is a constant coefficient operator
$$Ly = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0y$$
where each a_k is a constant, the associated or characteristic polynomial is the function P given by
$$P(r) = r^n + a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \cdots + a_1r + a_0$$
for all complex numbers r.

Definition. Saying that $r_1,r_2,\dots,r_l$ lists each zero of P exactly once and that r_i has multiplicity m_i for $i = 1,2,\dots,l$ means that
$$P(r) = (r-r_1)^{m_1}(r-r_2)^{m_2}\cdots(r-r_l)^{m_l},$$
where each r_i is a number, possibly complex, $r_i \neq r_j$ when $i \neq j$ and each m_i is a positive integer.

Theorem. When P is as above, a fundamental list for L is
$$\begin{array}{cccc} e^{r_1x} & xe^{r_1x} & \cdots & x^{m_1-1}e^{r_1x} \\ e^{r_2x} & xe^{r_2x} & \cdots & x^{m_2-1}e^{r_2x} \\ \vdots & \vdots & \cdots & \vdots \\ e^{r_lx} & xe^{r_lx} & \cdots & x^{m_l-1}e^{r_lx} \end{array}.$$
If $x^pe^{(\alpha+\beta i)x}$ and $x^pe^{(\alpha-\beta i)x}$ occur in this list, this pair can and should be replaced with $x^pe^{\alpha x}\cos\beta x$ and $x^pe^{\alpha x}\sin\beta x$.

Example. Find a fundamental list or set then find all solutions to
$$y''' - 6y'' + 11y' - 6y = 0.$$
Solution. The polynomial P is given by

$$P(r) = r^3 - 6r^2 + 11r - 6.$$

The sum of the coefficients ($1 - 6 + 11 - 6$) is zero so the number 1 is a zero of P. $P(1) = 0$. Long division or synthetic division shows that

$$\frac{P(r)}{r-1} = \frac{r^3 - 6r^2 + 11r - 6}{r-1} = r^2 - 5r + 6$$

so

$$P(r) = (r-1)(r^2 - 5r + 6) = (r-1)(r-2)(r-3).$$

The zeros of P are 1, 2, and 3 and each is of multiplicity one. A fundamental list or set is

$$\{e^x, e^{2x}, e^{3x}\}$$

and y is a solution to the DE if and only if

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

for some triple of numbers c_1, c_2, and c_3.

Example. Find a fundamental list or set then find all solutions to

$$y^{(4)} - y = 0.$$

Solution. The polynomial P is given by

$$P(r) = r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r-1)(r+1)(r^2 + 1).$$

The zeros of $(r^2 + 1)$ are $0 + i$ and $0 - i$. A fundamental list or set is

$$\{e^x, e^{-x}, \cos x, \sin x\},$$

and y is a solution to the DE if and only if

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

for some list of number c_1, c_2, c_3, and c_4.

Nonhomogeneous Equations.

In this part of Section 3.7, we will be concerned with finding the solutions to the nonhomogeneous equation

$$Ly = f \qquad \text{N}$$

where

$$Ly = y^{(n)} + p_{n-1} y^{(n-1)} + p_{n-2} y^{(n-2)} + \cdots + p_1 y' + p_0 y$$

on an interval J and each p_k and f is a continuous function with domain J. In order to solve (N) we will first need to solve the related homogeneous equation

$$Ly = 0 \qquad \text{H}$$

which is sometimes called the **reduced equation**..

Recall that

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2.$$

Consequently,

$$L(y_1 + y_2) = Ly_1 + Ly_2,\ L(y_1 - y_2) = Ly_1 - Ly_2, \text{ and } L(cy) = cLy.$$

Also, if

$$Ly_1 = f \text{ and } Ly_2 = f \text{ then } L(y_1 - y_2) = f - f = 0.$$

The difference of two solutions to (N) is a solution to (H).

$$Ly = f \qquad \text{N}$$

$$Ly = 0 \qquad \text{H}$$

In order to find all solutions to the nonhomogeneous equation (N) we need one solution of (N) (called a **particular solution**) and all solutions of the corresponding homogeneous or reduced equation (H).

Theorem. Suppose that

$$Lz = f \text{ on } J.$$

(The function z is called a **particular solution** to the nonhomogeneous equation (N), and the following is a description of all solutions to (N).) It follows that

$$Ly = f \text{ on } J \text{ if and only if}$$

$$y = u + z \text{ for some } u \text{ such that}$$

$$Lu = 0..$$

Proof. If $Ly = f$, let $u = y - z$. Then $y = u + z$ and $Lu = L(y - z) = Ly - Lz = f - f = 0$ on J.

If $y = u + z$ and $Lu = 0$ on J, then $Ly = L(u + z) = Lu + Lz = 0 + f = f$ on J.

If $\{y_1, y_2, \ldots, y_n\}$ is a fundamental list or set for L or (H), the u in the last theorem can be replaced with

$$c_1y_1 + c_2y_2 + \cdots c_ny_n$$

Theorem. Suppose that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental list or set for L, and

$$Lz = f \text{ on } J.$$

It follows that

$$Ly = f \text{ on } J \text{ if and only if}$$
$$y = c_1 y_1 + c_2 y_2 + \cdots c_n y_n + z \text{ for some list of numbers } c_1, \ldots, c_n.$$

Note. There is an extension of the Variation of Parameters formula that applies to higher order equations. A particular solution z satisfying $Lz = f$ is given by

$$z(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x) f(x)}{W(x)} dx$$

where $(y_1, \ldots, y_n)$ is a fundamental list for L, W is their Wronskian, and W_k is the determinant of the matrix obtained by replacing the k-th column of their Wronski matrix with

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

However, we will not need this formula for the problems in this section.

Note. The method of undetermined coefficients can be used to find a particular solution to

$$Ly = f$$

when L is constant coefficient and f is of the type where the method works for second order equations.

Additional Examples: See Section 3.7 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 3.7. The answers are posted on Dr. Walker's web site.