# Section 3.7
# Higher Order Linear Differential Equations

**Definition**. When $n$ is a positive integer, saying that $L$ is an

over an interval $J$ means that there is a list of continuous functions $(p_0, p_1, \ldots, p_{n-1})$ each defined on $J$ such that

whenever $y$ is an $n$-times differentiable function defined on $J$. We will be concerned with the **homogeneous** differential equation

1

the **nonhomogeneous** differential equation

2

and the initial value problems consisting of (1) or (2) and

where $x_0$ is a number in $J$ and each of $k_0, k_1, \ldots, k_{n-1}$ is a number.

**Theorem**. If $L$ is an $n^{th}$order linear differential operator over an interval $J$, each of $y_1$ and $y_2$ is an $n$-times differentiable function with domain $J$, and each of $c_1$ and $c_2$ is a number, then

Special cases are

and .

**Corollary**. If $L$ is an $n^{th}$ order linear differential operator over an interval $J$, $m$ is a positive integer, each of $y_1$, $y_2$, ... , $y_m$ is an $n$-times differentiable function with domain $J$, and each of $c_1$, $c_2$, ... , $c_m$ is a number, then

We will accept the following uniqueness and existence theorem and use it as a basis for developing a description of all solutions to the homogeneous equation. An indication of proof will be given in a later chapter

.

**Theorem**. Suppose that $L$ is an $n^{th}$ order linear differential operator over the interval $J$. If $x_0$ is a number in $J$ and each of $k_0, k_1, \ldots, k_{n-1}$ is a number, there is a unique function $y$ defined on $J$ such that

**Theorem**. Every linear combination of solutions to the homogeneous equation is also a solution.

If

each of $c_1$, $c_2$, … ,$c_m$ is a number and

then

This is true because

$=$

$=$

## Homogeneous Equations

**Definition**. Suppose that $y_k$ is a function with domain $J$ for $k = 1, \ldots, m$. Saying that $(y_1, \ldots, y_m)$ is a list of functions that are **linearly independent** over $J$ means that if each of $c_1, \ldots c_m$ is a number and

then

Saying that $(y_1, \ldots, y_m)$ is a list of functions that are **linearly dependent** over $J$ means that it is a list of functions that are

**Note**. $(y_1, \ldots, y_m)$ is a list of functions that are **linearly dependent** over $J$ means if and only if there is a list of numbers $c_1, \ldots, c_m$ such that

for all $x$ in $J$.

**Definition.** When $(y_1, \ldots, y_n)$ is a of functions each defined on an interval $J$ and each having $n-1$ derivatives, their **Wronski matrix** is given by

$$M_W[y_1, \ldots, y_n] =$$

and their **Wronskian** is given by

$$W[y_1, \ldots, y_n] =$$

**Theorem.** (**First Wronskian Test**) If for some number $x_0$ in the interval $J$, then $(y_1, \ldots, y_n)$ a list of functions that are linearly independent over $J$.

**Theorem** (**Second Wronskin Test**) and for some number $x_0$ in $J$, then $(y_1, \ldots, y_n)$ a list of functions that are linearly dependent over $J$.

**Definition.** Saying that $(y_1, \ldots, y_n)$ or $\{y_1, \ldots, y_n\}$ is a or for $L$ or for $Ly = 0$ means that

for $k = 1, \ldots n$

and

over $J$.

The following theorem gives a description of all solutions to the homogeneous equation.

**Theorem**. If $(y_1, \ldots, y_n)$ is a fundamental list for $L$ then

if and only if

for some list of numbers

$$(c_1, c_2, \ldots, c_n)$$

**Definition**. When $L$ is a constant coefficient operator

where , the associated or characteristic polynomial is the function $P$ given by

for all complex numbers $r$.

**Definition**. Saying that $r_1, r_2, \ldots, r_l$ lists each zero of $P$ exactly once and that $r_i$ has multiplicity $m_i$ for $i = 1, 2, \ldots, l$ means that

,

where each $r_i$ is a number, possibly complex, $r_i \neq r_j$ when $i \neq j$ and each $m_i$ is a positive integer.

**Theorem**. When $P$ is as above, a fundamental list for $L$ is

.

If $x^p e^{(\alpha+\beta i)x}$ and $x^p e^{(\alpha-\beta i)x}$ occur in this list, this pair can and should be replaced with and .

**Example**. Find a fundamental list or set then find all solutions to

**Solution**. The polynomial $P$ is given by

The sum of the coefficients ( ) is zero . .
Long division or synthetic division shows that

so

$$P(r) = \quad = \quad .$$

The zeros of $P$ are . A fundamental list or set is

and $y$ is a solution to the DE if and only if

for some triple of numbers $c_1$, $c_2$, and $c_3$.

**Example**. Find a fundamental list or set then find all solutions to

.

**Solution**. The polynomial $P$ is given by

$$P(r) = \quad = \quad =$$

The zeros of A fundamental list or set is

,

and $y$ is a solution to the DE if and only if

for some list of number $c_1$, $c_2$, $c_3$, and $c_4$.

**Nonhomogeneous Equations**.

In this part of Section 3.7, we will be concerned with finding the solutions to the nonhomogeneous equation

N

where

on an interval $J$ and each $p_k$ and $f$ is a continuous function with domain $J$. In order to solve (N) we will first need to solve the related homogeneous equation

H

which is sometimes called the .

Recall that

Consequently,

, and .

Also, if

$$Ly_1 = f \text{ and } Ly_2 = f \text{ then}$$

The is a solution to .

$$Ly = f \qquad \text{N}$$

$$Ly = 0 \qquad \text{H}$$

In order to find all solutions to the nonhomogeneous equation (N) we need and of the corresponding homogeneous or reduced equation (H).

**Theorem**. Suppose that

$$Lz = f \text{ on } J.$$

(The function $z$ is called a **particular solution** to the nonhomogeneous equation (N), and the following is a description of all solutions to (N).) It follows that

.

**Proof**. If $Ly = f$, let $u = y - z$. Then and on $J$.

If $y = u + z$ and $Lu = 0$ on $J$, then $f$ on $J$.

If $\{y_1, y_2, \ldots, y_n\}$ is a fundamental list or set for $L$ or (H), the $u$ in the last theorem can be replaced with

**Theorem**. Suppose that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental list or set for $L$, and

$$Lz = f \text{ on } J.$$

It follows that

**Note**. There is an extension of the Variation of Parameters formula that applies to higher order equations. A particular solution $z$ satisfying $Lz = f$ is given by

where $(y_1, \ldots, y_n)$ is a fundamental list for $L$, $W$ is their Wronskian, and $W_k$ is the determinant of the matrix obtained by replacing the $k$-th column of their Wronski matrix with

However, we will not need this formula for the problems in this section.

**Note**. The method of undetermined coefficients can be used to find a particular solution to

when and

**Additional Examples**: See Section 3.7 of the text and the notes presented on the board in class.

**Suggested Problems**. Do the odd numbered problems for Section 3.7. The answers are posted on Dr. Walker's web site.