Section 4.1
The Laplace Transform - Introduction

Definition. When g is integrable on $[a,b]$ for each $b \geq a$, saying that

$$\int_a^\infty g(x)dx$$

exists means that there is a number l such that

$$\lim_{b\to\infty} \int_a^b g(x)dx = l.$$

In this case

$$\int_a^\infty g(x)dx = l.$$

To find

$$\int_a^\infty g(x)dx$$

or show that it does not exist, first find

$$\int_a^b g(x)dx$$

and determine whether or not the limit as $b \to \infty$ exists. If the limit does exist, then

$$\int_a^\infty g(x)dx$$

is that limit.

Example.

$$\int_1^b \frac{1}{x^2}dx = -\left[\frac{1}{x}\right]_{x=1}^{x=b} = \frac{1}{1} - \frac{1}{b}$$

and

$$\lim_{b\to\infty}\left[\frac{1}{1} - \frac{1}{b}\right] = [1-0] = 1$$

so

$$\int_1^\infty \frac{1}{x^2}dx = 1$$

Example.

$$\int_1^b \frac{1}{x}dx = [\ln x]_{x=1}^{x=b} = \ln b \to \infty \text{ as } b \to \infty$$

so

$$\int_1^\infty \frac{1}{x}dx \text{ does not exist.}$$

Definition. Suppose that f is a function with domain $[0,\infty)$ which is integrable on $[0,b]$ for each $b > 0$ and

$$\int_0^\infty e^{-sx}f(x)dx$$

exists for some number s. The **Laplace** transform of f is the function F given by

$$F(s) = \int_0^\infty e^{-sx}f(x)dx$$

for all numbers s where the integral exists.

Theorem. If r is a real number and

$$f(x) = e^{rx}$$

for $x \geq 0$, then the Laplace transform of f is F where

$$F(s) = \frac{1}{s-r}$$

for $s > r$.

Proof.

$$\int_0^b e^{-sx}e^{rx}dx = \int_0^b e^{(r-s)x}dx = \begin{cases} \int_0^b 1dx = b & if \quad s = r \\ \frac{1}{r-s}[e^{(r-s)x}]_{x=0}^{x=b} & if \quad s \neq r \end{cases}$$

so

$$\int_0^b e^{-sx}e^{rx}dx = b \text{ if } s = r \text{ and}$$

$$\int_0^b e^{-sx}e^{rx}dx = \frac{1}{r-s}[e^{(r-s)b} - e^0] \text{ if } s \neq r.$$

So

$$\lim_{b\to\infty} \int_0^b e^{-sx}e^{rx}dx = \begin{cases} \infty & if \quad r-s \geq 0 \\ \frac{1}{r-s}[0-1] = \frac{1}{s-r} & if \quad r-s < 0 \end{cases}.$$

Note that $r-s < 0$ is equivalent to $s > r$.

Definition. The Laplace transform of f will be denoted by $\mathcal{L}f$ and $\mathcal{L}\{\text{formula for } f(x)\}(s)$ will denote the value at s of the Laplace transform of the function f whose formula is given by "the formula for $f(x)$." We may also write $\mathcal{L}[\text{formula for } f(x)]$ in place of $\mathcal{L}\{\text{formula for } f(x)\}(s)$, interperting s as the identity function

Thus

$$\mathcal{L}\{e^{rx}\}(s) = \frac{1}{s-r} \text{ for } s > r.$$

or

$$\mathcal{L}[e^{rx}] = \frac{1}{s-r} \text{ for } s > r.$$

Corollary.

$$\mathcal{L}\{1\}(s) = \frac{1}{s} \text{ for } s > 0.$$

Proof. This follows from $\mathcal{L}\{e^{rx}\}(s) = \frac{1}{s-r}$ for $s > r$ because $e^{0x} = 1$.

Theorem.

$$\mathcal{L}\{x\}(s) = \frac{1}{s^2} \text{ for } s > 0.$$

Proof.

$$\begin{aligned}\int_0^b e^{-sx}x\,dx &= [-\frac{1}{s}e^{-sx}x]_{x=0}^{x=b} + \int_0^b \frac{1}{s}e^{-sx} \cdot 1\,dx \\ &= -\frac{1}{s}be^{-sb} - \frac{1}{s^2}e^{-sb} + \frac{1}{s^2}e^0\end{aligned}$$

so

$$\lim_{b\to\infty} \int_0^b e^{-sx}x\,dx = -0 - 0 + \frac{1}{s^2} \cdot 1 = \frac{1}{s^2} \text{ when } s > 0.$$

Theorem.

$$\mathcal{L}\{x^2\}(s) = \frac{2}{s^3} \text{ for } s > 0.$$

Proof.

$$\begin{aligned}\int_0^b e^{-sx}x^2\,dx &= [-\frac{1}{s}e^{-sx}x^2]_{x=0}^{x=b} + \int_0^b \frac{1}{s}e^{-sx} \cdot 2x\,dx \\ &= -\frac{1}{s}b^2e^{-sb} + \frac{2}{s}\int_0^b e^{-sx}x\,dx \to 0 + \frac{2}{s}\mathcal{L}\{x\}(s) \\ &= \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3} \text{ as } b \to \infty.\end{aligned}$$

Theorem.

$$\mathcal{L}\{x^3\}(s) = \frac{3!}{s^4} \text{ for } s > 0.$$

Proof.

$$\int_0^b e^{-sx}x^3dx = [-\frac{1}{s}e^{-sx}x^3]_{x=0}^{x=b} + \int_0^b \frac{1}{s}e^{-sx} \cdot 3x^2dx$$
$$= -\frac{1}{s}b^2e^{-sb} + \frac{3}{s}\int_0^b e^{-sx}x^2dx \to 0 + \frac{3}{s}\mathcal{L}\{x^2\}(s)$$
$$= \frac{3}{s}\frac{2}{s^3} = \frac{3!}{s^4} \text{ as } b \to \infty.$$

Theorem. When n is a positive integer

$$\mathcal{L}\{x^n\}(s) = \frac{n!}{s^{n+1}} \text{ for } s > 0.$$

Theorem. The Laplace transform is linear. If

$$\mathcal{L}\{f_1(x)\}(s) = F_1(s) \text{ and } \mathcal{L}\{f_2(x)\}(s) = F_2(s) \text{ for } s > s_0$$

and each of c_1 and c_2 is a number, then

$$\mathcal{L}\{c_1f_1(x) + c_2f_2(x)\}(s) = c_1F_1(s) + c_2F_2(s) \text{ for } s > s_0.$$

Proof. This follows because

$$\int_0^b e^{-sx}(c_1f_1(x) + c_2f_2(x))dx = c_1\int_0^b e^{-sx}f_1(x)dx + c_2\int_0^b e^{-sx}f_2(x)dx$$

for each $s > s_0$.

This extends to

$$\mathcal{L}\{c_1f_1(x) + \cdots + c_nf_n(x)\}(s) = c_1\mathcal{L}\{f_1(x)\}(s) + \cdots + c_n\mathcal{L}\{f_n(x)\}(s).$$

$$\mathcal{L}[2e^{3x} - 5x^2 + 3] = 2\frac{1}{s-3} - 5\frac{2}{s^3} + 3\frac{1}{s} = \frac{2}{s-3} - \frac{10}{s^3} + \frac{3}{s}$$

Definition. When θ is a real number

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

consequently

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta, \\ e^{-i\theta} &= \cos\theta - i\sin\theta, \\ \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \text{ and} \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \end{aligned}$$

The formula

$$\mathcal{L}\{e^{rx}\}(s) = \frac{1}{s-r}$$

remains valid when r is complex provided that $s > \operatorname{Re} r$.

When $r = \alpha + \beta i$ with each of α and β real, $\operatorname{Re} r = \alpha$. $\operatorname{Re} r$ is called the real part of r and $\operatorname{Im} r = \beta$. $\operatorname{Im} r$ is called the imaginary part of r.

Theorem.

$$\mathcal{L}\{\cos\beta x\}(s) = \frac{s}{s^2 + \beta^2} \text{ for } s > 0.$$

Proof.

$$\begin{aligned} \mathcal{L}\{\cos\beta x\}(s) &= \mathcal{L}\left\{\frac{1}{2}(e^{i\beta x} + e^{-i\beta x})\right\}(s) = \mathcal{L}\left\{\frac{1}{2}e^{i\beta x} + \frac{1}{2}e^{-i\beta x}\right\}(s) \\ &= \frac{1}{2}\mathcal{L}\{e^{i\beta x}\}(s) + \frac{1}{2}\mathcal{L}\{e^{-i\beta x}\}(s) \\ &= \frac{1}{2}\frac{1}{s-i\beta} + \frac{1}{2}\frac{1}{s+i\beta} \\ &= \frac{1}{2}\frac{(s+i\beta)+(s-i\beta)}{(s-i\beta)(s+i\beta)} \\ &= \frac{1}{2}\frac{2s}{s^2-(i\beta)^2} \\ &= \frac{s}{s^2+\beta^2} \end{aligned}$$

for $s > 0$.

Theorem.

$$\mathcal{L}\{\sin\beta x\}(s) = \frac{\beta}{s^2 + \beta^2} \text{ for } s > 0.$$

Proof.

$$\begin{aligned}
\mathcal{L}\{\sin\beta x\}(s) &= \mathcal{L}\left\{\frac{1}{2i}(e^{i\beta x} - e^{-i\beta x})\right\}(s) = \mathcal{L}\left\{\frac{1}{2i}e^{i\beta x} - \frac{1}{2i}e^{-i\beta x}\right\}(s) \\
&= \frac{1}{2i}\mathcal{L}\{e^{i\beta x}\}(s) - \frac{1}{2i}\mathcal{L}\{e^{-i\beta x}\}(s) \\
&= \frac{1}{2i}\frac{1}{s - i\beta} - \frac{1}{2i}\frac{1}{s + i\beta} \\
&= \frac{1}{2i}\frac{(s + i\beta) - (s - i\beta)}{(s - i\beta)(s + i\beta)} \\
&= \frac{1}{2i}\frac{2i\beta}{s^2 - (i\beta)^2} \\
&= \frac{\beta}{s^2 + \beta^2}
\end{aligned}$$

for $s > 0$.

Suggested Problems. Do problems 1-8 for Section 4.1.
Note that

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

and

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

Additional Examples: See Section 4.1 of the text and the notes presented on the board in class.