Section 4.1

Section 4.1
The Laplace Transform - Introduction

Definition. When g is integrable on $[a,b]$ for each $b \geq a$, saying that

exists means that there is a number l such that

In this case

To find

or show that it does not exist, first find

and determine whether or not the limit as $b \to \infty$ exists. If the limit does exist, then

is that limit.

Example.

$$\int_1^b \frac{1}{x^2}\,dx = \qquad = $$

and

$$= \qquad =$$

so

Example.

$$\int_1^b \frac{1}{x}\,dx = \qquad = \qquad$$

so

Definition. Suppose that f is a function with domain $[0,\infty)$ which is integrable on $[0,b]$ for each $b > 0$ and

exists for some number s. The **Laplace** transform of f is the function F given by

for all numbers s where the integral exists.

Theorem. If r is a real number and

for $x \geq 0,$ then the Laplace transform of f is F where

for $s > r$.

Proof.

$$\int_0^b e^{-sx}e^{rx}\,dx = \qquad = \begin{cases} \\ \end{cases}$$

so

.

So

$$\lim_{b\to\infty}\int_0^b e^{-sx}e^{rx}\,dx = \begin{cases} \\ \end{cases} .$$

Note that $r - s < 0$ is equivalent to $s > r$.

Definition. The Laplace transform of f will be denoted by and will denote the value at s of the Laplace transform of the function f whose formula is given by "the formula for $f(x)$." We may also write in place of , interperting s as the identity function

Thus

.

or

Corollary.

Proof. This follows from $\mathcal{L}\{e^{rx}\}(s) = \frac{1}{s-r}$ for $s > r$ because $e^{0x} = 1$.

Theorem.

$$\mathcal{L}\{x\}(s) = \quad .$$

Proof.

$$\int_0^b e^{-sx} x dx = \quad + \quad .$$

$$=$$

so

$$\lim_{b\to\infty} \int_0^b e^{-sx} x dx =$$

Theorem.

$$\mathcal{L}\{x^2\}(s) = \quad .$$

Proof.

$$\int_0^b e^{-sx}x^2 dx = [-\tfrac{1}{s}e^{-sx}x^2]_{x=0}^{x=b} + \int_0^b \tfrac{1}{s}e^{-sx} \cdot 2x dx$$

$$= -\tfrac{1}{s}b^2e^{-sb} + \tfrac{2}{s}\int_0^b e^{-sx} x dx \to 0 + \tfrac{2}{s}\mathcal{L}\{x\}(s)$$

$$= \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3} \text{ as } b \to \infty.$$

Theorem.

$$\mathcal{L}\{x^3\}(s) = \frac{3!}{s^4} \text{ for } s > 0.$$

Proof.

$$\int_0^b e^{-sx} x^3 dx =$$

$$= \qquad \rightarrow$$

$$= \qquad = \qquad \text{as } b \rightarrow \infty.$$

Theorem. When n is a positive integer

$$\mathcal{L}\{x^n\}(s) =$$

Theorem. The Laplace transform is linear. If

$$\mathcal{L}\{f_1(x)\}(s) = F_1(s) \text{ and } \mathcal{L}\{f_2(x)\}(s) = F_2(s) \text{ for } s > s_0$$

and each of c_1 and c_2 is a number, then

Proof. This follows because

for each $s > s_0$.

This extends to

$$\mathcal{L}[2e^{3x} - 5x^2 + 3] = \qquad =$$

Definition. When θ is a real number

$$e^{i\theta} =$$

so

$$e^{-i\theta} = \qquad =$$

consequently

$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{-i\theta} = \cos\theta - i\sin\theta,$$
$$\cos\theta = \quad , \text{ and}$$
$$\sin\theta = \quad).$$

The formula

$$\mathcal{L}\{e^{rx}\}(s) =$$

remains valid when r is complex provided that $s > \operatorname{Re} r$.

When $r = \alpha + \beta i$ with each of α and β real, $\operatorname{Re} r = \alpha$. $\operatorname{Re} r$ is called the real part of r and $\operatorname{Im} r = \beta$. $\operatorname{Im} r$ is called the imaginary part of r.

Theorem.

$$\mathcal{L}\{\cos\beta x\}(s) =$$

Proof.

$$\mathcal{L}\{\cos\beta x\}(s) = \quad = $$
$$=$$
$$= \quad + $$
$$=$$
$$=$$
$$=$$

for $s > 0$.

Theorem.

$$\mathcal{L}\{\sin\beta x\}(s) = \quad .$$

Proof.

$$\mathcal{L}\{\sin\beta x\}(s) =$$

for $s > 0$.

Suggested Problems. Do problems 1-8 for Section 4.1.
Note that

$$\cosh x =$$

and

$$\sinh x =$$

Additional Examples: See Section 4.1 of the text and the notes presented on the board in class.