Section 4.2

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The Laplace Transform - Basic Properties

Definition. Suppose that f is a function with domain $[0,\infty)$and λ is a real number. Saying that f is of exponential order λ means that there is a nonnegative number M and a nonnegative number A such that

$$|f(x)| \leq Me^{\lambda x} \text{ for all } x \geq A.$$

Saying that f is of exponential order means that f is of exponential order λ for some number λ.

Note. If $f(x) =$

$$p(x),$$
$$p(x)e^{rx},$$
$$p(x)e^{rx}\cos\beta x, \text{ or } p(x)e^{rx}\sin\beta x$$

where p is a polynomial and each of r and β is a real number, then f is of exponential order.

Example. If

$$f(x) = e^{x^2}$$

then f is not of exponential order. $f(x)$ grows too fast as $x \to \infty$.

Theorem. If f is continuous and of exponential order λ, then

$$\mathcal{L}\{f(x)\}(s)$$

exists for all $s > \lambda$.

Theorem. The Laplace transform is linear. If

$$\mathcal{L}\{f_1(x)\}(s) = F_1(s) \text{ and } \mathcal{L}\{f_2(x)\}(s) = F_2(s) \text{ for } s > s_0$$

and each of c_1 and c_2 is a number, then

$$\mathcal{L}\{c_1f_1(x) + c_2f_2(x)\}(s) = c_1F_1(s) + c_2F_2(s) \text{ for } s > s_0.$$

Proof. This follows because

$$\int_0^b e^{-sx}(c_1f_1(x) + c_2f_2(x))dx = c_1\int_0^b e^{-sx}f_1(x)dx + c_2\int_0^b e^{-sx}f_2(x)dx$$

for each $s > s_0$.

This extends to

$$\mathcal{L}\{c_1 f_1(x) + \cdots + c_n f_n(x)\}(s) = c_1\mathcal{L}\{f_1(x)\}(s) + \cdots + c_n\mathcal{L}\{f_n(x)\}(s).$$

Theorem. If y is of exponential order λ and has a continuous derivative on $[0,\infty)$ then

$$\mathcal{L}\{y'(x)\}(s) = s\mathcal{L}\{y(x)\}(s) - y(0)$$

for all $s > \lambda$.

Proof.

$$\begin{aligned}\int_0^b e^{-sx}y'(x)dx &= [e^{-sx}y(x)]_{x=0}^{x=b} + s\int_0^b e^{-sx}y(x)dx \\ &= e^{-sb}y(b) - e^0y(0) + s\int_0^b e^{-sx}y(x)dx \\ &= e^{-sb}y(b) + s\int_0^b e^{-sx}y(x)dx - y(0) \\ &\to 0 + s\mathcal{L}\{y(x)\}(s) - y(0) \text{ as } b \to \infty.\end{aligned}$$

when $s > \lambda$.

Corollary. If y has a continuous n-th derivative and each of y, y', … , and $y^{(n-1)}$ is or exponential order λ, then

$$\mathcal{L}\{y^{(n)}(x)\}(s) = s^n\mathcal{L}\{y(x)\}(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

Note. If

$$Y(s) = \mathcal{L}\{y(x)\}(s)$$

then

$$\mathcal{L}\{y'(x)\}(s) = sY(s) - y(0)$$

and

$$\mathcal{L}\{y''(x)\}(s) = s^2Y(s) - sy(0) - y'(0).$$

Example. Find the Laplace transform of the solution to the following initial value problem.

$$y'(x) + 5y(x) = x \text{ for } x \geq 0 \text{ and } y(0) = 3.$$

Solution. Taking the Laplace transform of each side of the differential equation and using the fact that the transform is linear we have

$$\mathcal{L}\{y'(x)\}(s) + 5\mathcal{L}\{y(x)\}(s) = \mathcal{L}\{x\}(s)$$

Letting

$$Y(s) = \mathcal{L}\{y(x)\}(s),$$

and using

$$\mathcal{L}\{y'(x)\}(s) = sY(s) - y(0)\ ,\ \mathcal{L}\{x\}(s) = \frac{1}{s^2}, \text{ and } y(0) = 3$$

we have

$$sY(s) - 3 + 5Y(s) = \frac{1}{s^2}.$$

$$sY(s) - 3 + 5Y(s) = \frac{1}{s^2}.$$

So

$$(s+5)Y(s) = 3 + \frac{1}{s^2}.$$

Thus

$$Y(s) = \frac{3}{s+5} + \frac{1}{(s+5)s^2}.$$

Example. Find the Laplace transform of the solution to the following initial value problem.

$$y''(x) - y'(x) + 6y(x) = \sin 2x \text{ for } x \geq 0,\ y(0) = -1, \text{ and } y'(0) = 4.$$

Solution. Letting $Y(s) = \mathcal{L}\{y(x)\}(s)$ and taking the Laplace transform of each side of the differential equation we have

$$(s^2Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) + 6Y(s) = \frac{2}{s^2+4}$$

so

$$(s^2Y(s) - s(-1) - 4) - (sY(s) - (-1)) + 6Y(s) = \frac{2}{s^2+4}$$

so

$$(s^2 - s + 6)Y(s) + s - 5 = \frac{2}{s^2+4}.$$

From

$$(s^2 - s + 6)Y(s) + s - 5 = \frac{2}{s^2+4}$$

we have

$$Y(s) = \frac{5-s}{(s^2-s+6)} + \frac{2}{(s^2-s+6)(s^2+4)}.$$

The next theorem shows how to find the Laplace transform of $e^{rx}f(x)$ if the transform of $f(x)$ can be found.

Theorem. If

$$\mathcal{L}\{f(x)\}(s) = F(s) \text{ for } s > s_0$$

then

$$e^{rx}f(x) = F(s-r) \text{ for } s > s_0 + r$$

Example.

$$\mathcal{L}\{\cos 3x\}(s) = \frac{s}{s^2+9};$$

so

$$\mathcal{L}\{e^{2x}\cos 3x\}(s) = \frac{s-2}{(s-2)^2+9}.$$

The next theorem shows how to find the Laplace transform of $x^n f(x)$ if the transform of $f(x)$can be found.

Theorem. If

$$\mathcal{L}\{f(x)\}(s) = F(s)$$

then

$$\mathcal{L}\{x^n f(x)\}(s) = (-1)^n F^{(n)}(s).$$

Example.

$$\mathcal{L}\{\sin 3x\}(s) = F(s)$$

where

$$F(s) = \frac{3}{s^2+9}.$$

$$F'(s) = \frac{-6s}{(s^2+9)^2}$$

and

$$F''(s) = \cdots = \frac{18s^2-54}{(s^2+9)^3}$$

so

$$\mathcal{L}\{x^2 \sin 3x\}(s) = \frac{18s^2 - 54}{(s^2+9)^3}$$

Additional Examples: See Section 4.2 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 4.2. The answers are posted on Dr. Walker's web site.