# Section 4.2

Section 4.2

## The Laplace Transform - Basic Properties

**Definition.** Suppose that $f$ is a function with domain $[0, \infty)$ and $\lambda$ is a real number. Saying that means that there is a and a such that

for all $x \geq A$.

Saying that $f$ is of exponential order means that $f$ is of exponential order $\lambda$ for some number $\lambda$.

**Note.** If $f(x) =$

or

where and each of $r$ and $\beta$ is a real number, then $f$ is of exponential order.

**Example.** If

then $f$ is not of exponential order. $f(x)$ grows too fast as $x \to \infty$.

**Theorem.** , then

$$\mathcal{L}\{f(x)\}(s)$$

exists for all $s > \lambda$.

**Theorem**. If $y$ is of exponential order $\lambda$ and has a continuous derivative on $[0, \infty)$ then

$$\mathcal{L}\{y'(x)\}(s) = \boxed{\qquad\qquad}$$

for all $s > \lambda$.

**Proof**.

$$\begin{aligned} \int_0^b e^{-sx} y'(x)dx &= \boxed{\qquad\qquad} \\ &= \boxed{\qquad\qquad} \\ &= \boxed{\qquad\qquad} \\ &\to \boxed{\qquad\qquad} \text{ as } b \to \infty. \end{aligned}$$

when $s > \lambda$.

**Corollary**. If $y$ has a continuous $n$-th derivative and each of $y$, $y'$, … , and $y^{(n-1)}$ is or exponential order $\lambda$, then

$$\mathcal{L}\{y^{(n)}(x)\}(s) = \boxed{\qquad\qquad\qquad\qquad}$$

**Note**. If

$$Y(s) = \mathcal{L}\{y(x)\}(s)$$

then

$$\mathcal{L}\{y'(x)\}(s) = \boxed{\qquad}$$

and

$$\mathcal{L}\{y''(x)\}(s) = \boxed{\qquad\qquad}.$$

**Example**. Find the Laplace transform of the solution to the following initial value problem.

$$\boxed{\qquad\qquad} \text{ for } x \geq 0 \text{ and } \boxed{\qquad}.$$

**Solution**. Taking the Laplace transform of each side of the differential equation and using the fact that the transform is linear we have

Letting

$$Y(s) =$$

and using

$$\mathcal{L}\{y'(x)\}(s) = \quad , \quad \mathcal{L}\{x\}(s) =$$

we have

So

Thus

$$Y(s) =$$

**Example**. Find the Laplace transform of the solution to the following initial value problem.

for $x \geq 0$, , and .

**Solution**. Letting $Y(s) = \mathcal{L}\{y(x)\}(s)$ and taking the Laplace transform of each side of the differential equation we have

so

so

From

we have

The next theorem shows how to find the Laplace transform of $e^{rx}f(x)$ if the transform of $f(x)$ can be found.

**Theorem**. If

$$\mathcal{L}\{f(x)\}(s) = F(s) \text{ for } s > s_0$$

then

$$\mathcal{L}\{e^{rx}f(x)\} = F(s-r) \text{ for } s > s_0 + r$$

**Example**.

$$\mathcal{L}\{\cos 3x\}(s) = \quad ;$$

so

$$\mathcal{L}\{e^{2x}\cos 3x\}(s) = \quad .$$

The next theorem shows how to find the Laplace transform of $x^n f(x)$ if the transform of $f(x)$can be found.

**Theorem**. If

$$\mathcal{L}\{f(x)\}(s) = F(s)$$

then

$$\mathcal{L}\{x^n f(x)\}(s) =$$

**Example**.

$$\mathcal{L}\{\sin 3x\}(s) = F(s)$$

where

$$F(s) = \quad .$$

$$F'(s) =$$

and

$$F''(s) = \cdots =$$

so

$$\mathcal{L}\{x^2 \sin 3x\}(s) = \boxed{\phantom{xxxx}}$$

**Additional Examples**: See Section 4.2 of the text and the notes presented on the board in class.

**Suggested Problems**. Do the odd numbered problems for Section 4.2. The answers are posted on Dr. Walker's web site.