Section 4.3

Definition. The inverse Laplace transform is denoted by $\mathcal{L}^{-1}$.

$$\mathcal{L}^{-1}\{F(s)\}(x) = f(x) \text{ means } \mathcal{L}\{f(x)\}(s) = F(s).$$

Note. Based on the transforms that we know, we have the following.

$$\mathcal{L}^{-1}\{\frac{1}{s-r}\}(x) = e^{rx}$$

$$\mathcal{L}^{-1}\{\frac{1}{s}\}(x) = 1$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\}(x) = x$$

$$\mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\}(x) = x^n$$

$$\mathcal{L}^{-1}\{\frac{\beta}{s^2+\beta^2}\}(x) = \sin\beta x$$

$$\mathcal{L}^{-1}\{\frac{s}{s^2+\beta^2}\}(x) = \cos\beta x$$

$$\mathcal{L}^{-1}\{F(s-r)\}(x) = e^{rx}\mathcal{L}^{-1}\{F(s)\}(x)$$

Note. Since the transform is linear, the inverse transform is also linear.

Note. In order to find $\mathcal{L}^{-1}\{F(s)\}$ when F is a proper rational function with a quadratic factor in the denominator, factor the quadratic if it has real zeros, otherwise complete the square.

Example.

$$\frac{1}{s^2+7s+12} = \frac{1}{(s+3)(s+4)}$$

Using partial fractions we have

$$\frac{1}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

Adding the fractions, we have

$$\frac{1}{(s+3)(s+4)} = \frac{A(s+4)+B(s+3)}{(s+3)(s+4)}$$

Equating the numerators, we have

$$1 = A(s+4)+B(s+3) \qquad 1$$

Letting $s = -3$ in (1), we have

$$1 = A \cdot 1 \text{ so } A = 1.$$

Letting $s = -4$ in (1), we have

$$1 = B \cdot (-1) \text{ so } B = -1.$$

Thus

$$\frac{1}{(s+3)(s+4)} = \frac{1}{s+3} - \frac{1}{s+4}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{(s+3)(s+4)}\}(x) &= \mathcal{L}^{-1}\{\frac{1}{s+3}\}(x) - \mathcal{L}^{-1}\{\frac{1}{s+4}\}(x)\\ &= e^{-3x} - e^{-4x}.\end{aligned}$$

Example.

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s+3}{s^2+4s+13}\}(x) &= \mathcal{L}^{-1}\{\frac{s+3}{(s+2)^2+9}\}(x)\\ &= \mathcal{L}^{-1}\{\frac{(s+2)+1}{(s+2)^2+9}\}(x)\\ &= e^{-2x}\mathcal{L}^{-1}\{\frac{s+1}{s^2+9}\}(x)\\ &= e^{-2x}\mathcal{L}^{-1}\{\frac{s}{s^2+9}\}(x) + e^{-2x}\mathcal{L}^{-1}\{\frac{1}{s^2+9}\}(x)\\ &= e^{-2x}\mathcal{L}^{-1}\{\frac{s}{s^2+9}\}(x) + \frac{1}{3}e^{-2x}\mathcal{L}^{-1}\{\frac{3}{s^2+9}\}(x)\\ &= e^{-2x}\cos 3x + \frac{1}{3}e^{-2x}\sin 3x\end{aligned}$$

Example. Use the Laplace transform to solve the initial value problem:

$$y'(x) + 2y(x) = 3\sin 4x \text{ for } x \geq 0 \text{ and } y(0) = 5.$$

Solution. Taking the Laplace transform of each side of the differential equation and letting $Y(s) = \mathcal{L}\{y(x)\}(s)$, we have

$$sY(s) - y(0) + 2Y(s) = 3\frac{4}{s^2+16}$$

so

$$(s+2)Y(s) - 5 = \frac{12}{s^2+16}.$$

So

$$Y(s) = \frac{5}{s+2} + \frac{12}{(s+2)(s^2+16)}$$

so

$$Y(s) = \frac{5s^2+92}{(s+2)(s^2+16)}.$$

Using partial fractions, we have

$$\frac{5s^2+92}{(s+2)(s^2+16)} = \frac{A}{s+2} + \frac{B+Cs}{s^2+16} = \frac{A(s^2+16)+(B+Cs)(s+2)}{(s+2)(s^2+16)}.$$

Equating numerators, we have

$$5s^2 + 92 = A(s^2 + 16) + (B + Cs)(s + 2) \qquad 2$$

Letting $s = -2$ in (2) produces

$$112 = 20A$$

so

$$A = \frac{112}{20} = \frac{28}{5}.$$

Letting $s = 0$ in (2) now produces

$$92 = \frac{28}{5} \cdot 16 + 2B$$

so

$$B = \frac{6}{5}.$$

Letting $s = 1$ in (2) now produces

$$97 = \frac{28}{5} \cdot 17 + (\frac{6}{5} + C)(3)$$

so

$$C = -\frac{3}{5}$$

$$\begin{aligned} Y(s) &= \frac{28/5}{s+2} + \frac{6/5 - 3/5s}{s^2 + 16} \\ &= \frac{28}{5}\frac{1}{s+2} + \frac{6}{5}\frac{1}{s^2+16} - \frac{3}{5}\frac{s}{s^2+16} \\ &= \frac{28}{5}\frac{1}{s+2} + \frac{6}{5}\frac{1}{4}\frac{4}{s^2+16} - \frac{3}{5}\frac{s}{s^2+16} \\ &= \frac{28}{5}\frac{1}{s+2} + \frac{3}{10}\frac{4}{s^2+4^2} - \frac{3}{5}\frac{s}{s^2+4^2}. \end{aligned}$$

So

$$y(x) = \frac{28}{5}e^{-2x} + \frac{3}{10}\sin 4x - \frac{3}{5}\cos 4x$$

Example. Use the Laplace transform to solve the initial value problem:

$$y''(x) + 4y'(x) + 6y(x) = 1 + e^{-x} \text{ for } x \geq 0,$$
$$y(0) = 0, \text{ and } y'(0) = 0.$$

Solution. Taking the Laplace transform of each side of the differential equation and letting $Y(s) = \mathcal{L}\{y(x)\}(s)$, we have

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$
$$(s^2 + 4s + 6)Y(s) = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}.$$

Using partial fractions, we find that

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{(s+1)} - \frac{s/2 + 5/3}{s^2 + 4s + 6}$$

so

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{(s+1)} - \frac{s/2 + 5/3}{(s+2)^2 + 2}.$$

Continuing we find that

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{(s+1)} - \frac{(s+2)/2 + 2/3}{(s+2)^2 + 2}$$

so

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{(s+1)} - \frac{1}{2}\frac{(s+2)}{(s+2)^2 + (\sqrt{2})^2} - \frac{2}{3\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2 + (\sqrt{2})^2}.$$

Thus

$$y(x) = \frac{1}{6} + \frac{1}{3}e^{-x} - \frac{1}{2}e^{-2s}\cos\sqrt{2}\,x - \frac{2}{3\sqrt{2}}e^{-2x}\sin\sqrt{2}\,x$$

Additional Examples: See Section 4.3 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 4.3. The answers are posted on Dr. Walker's web site.