Section 4.4

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Piecewise Defined and Piecewise Continuous Functions

Part I - Laplace Transforms

For the remainder of this chapter u will denote the unit step function, also known as the Heaviside function.

$$u(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}.$$

Note that if a is a number, then

$$u(x-a) = \begin{cases} 0 & if \quad x < a \\ 1 & if \quad x \geq a \end{cases}.$$

The unit step function can be used to write a piecewise defined function on one line.

Example. Suppose that

$$f(x) = \begin{cases} g_1(x) & if \quad x < a \\ g_2(x) & if \quad x \geq a \end{cases}.$$

Then

$$f(x) = g_1(x) + u(x-a)(-g_1(x) + g_2(x)).$$

Think in terms of the graph, moving from left to right along the x-axis. Start with g_1. When you get to a, turn off g_1 and turn on g_2.

Example. Suppose that

$$f(x) = \begin{cases} g_1(x) & if \quad x < a \\ g_2(x) & if \quad a \leq x < b \\ g_3(x) & if \quad b \leq x < c \\ g_4(x) & if \quad c \leq x \end{cases}.$$

Then

$$f(x) = g_1(x) + u(x-a)(-g_1(x) + g_2(x)) + u(x-b)(-g_2(x) + g_3(x)) \\ + u(x-c)(-g_3(x) + g_4(x))$$

From the definition of the Laplace transform, we have the fallowing results.

Theorem. If $a \geq 0$, then

$$\mathcal{L}\{u(x-a)\}(s) = \frac{e^{-as}}{s}.$$

Theorem. If $a \geq 0$, then

$$\mathcal{L}\{u(x-a)f(x-a)\}(s) = e^{-as}\mathcal{L}\{f(x)\}(s).$$

Example. Find $\mathcal{L}\{|2x-1|\}(s)$.

Solution.

$$|2x-1| = \begin{cases} 1-2x & if \;\; x < \frac{1}{2} \\ 2x-1 & if \;\; x \geq \frac{1}{2} \end{cases}$$

so

$$\begin{aligned} |2x-1| &= 1-2x+u(x-\frac{1}{2})(-(1-2x)+2x-1) \\ &= 1-2x+u(x-\frac{1}{2})(4x-2) \end{aligned}$$

so

$$|2x-1| = 1-2x+4u(x-\frac{1}{2})(x-\frac{1}{2})$$

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so

$$\mathcal{L}\{|2x-1|\}(s) = \frac{1}{s} - \frac{2}{s^2} + 4e^{-\frac{1}{2}s}\mathcal{L}\{x\}(s)$$

so

$$\mathcal{L}\{|2x-1|\}(s) = \frac{1}{s} - \frac{2}{s^2} + 4\frac{e^{-\frac{1}{2}s}}{s^2}$$

Example. Find the Laplace transform of f when

$$f(x) = \begin{cases} 2x-3 & if \;\; 0 \leq x < 3 \\ 3 & if \;\; 3 \leq x \end{cases}$$

Solution.

$$\begin{aligned} f(x) &= 2x - 3 + u(x-3)(-(2x-3)+3) \\ &= 2x - 3 - u(x-3)(2x-6) \\ & \, 2x - 3 - 2u(x-3)(x-3) \end{aligned}$$

so

$$\begin{aligned} \mathcal{L}\{f(x)\}(s) &= \frac{2}{s^2} - \frac{3}{s} - 2e^{-3s}\mathcal{L}\{x\}(s) \\ &= \frac{2}{s^2} - \frac{3}{s} - \frac{2e^{-3s}}{s^2}. \end{aligned}$$

Example. Find the Laplace transform of f when

$$f(x) = \begin{cases} -3 & if \quad 0 \le x < 4 \\ x & if \quad 4 \le x < 6 \\ -2 & if \quad 6 \le x \end{cases}.$$

Solution.

$$\begin{aligned} f(x) &= -3 + u(x-4)(-(-3)+x) + u(x-6)(-x-2) \\ &= -3 + u(x-4)(x+3) - u(x-6)(x+2) \\ &= -3 + u(x-4)((x-4)+7) - u(x-6)((x-6)+8) \\ &= -3 + u(x-4)(x-4) + 7u(x-4) \\ &\quad - u(x-6)(x-6) - 8u(x-6) \end{aligned}$$

$$\begin{aligned} f(x) &= -3 + u(x-4)(-(-3)+x) + u(x-6)(-x-2) \\ &= -3 + u(x-4)(x+3) - u(x-6)(x+2) \\ &= -3 + u(x-4)((x-4)+7) - u(x-6)((x-6)+8) \\ &= -3 + u(x-4)(x-4) + 7u(x-4) \\ &\quad - u(x-6)(x-6) - 8u(x-6) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{f(x)\}(s) &= \frac{-3}{s} + e^{-4s}\mathcal{L}\{x\}(s) + \frac{7e^{-4s}}{s} - e^{-6s}\mathcal{L}\{x\}(s) - \frac{8e^{-4s}}{s} \\ &= \frac{-3}{s} + e^{-4s}\frac{1}{s^2} + \frac{7e^{-4s}}{s} - e^{-6s}\frac{1}{s^2} - \frac{8e^{-6s}}{s} \end{aligned}$$

Example. Find the Laplace transform of f when

$$f(x) = \begin{cases} -4 & if \quad 0 \le x < \pi \\ 3\sin 2x & if \quad \pi \le x \end{cases}.$$

Solution.

$$
\begin{aligned}
f(x) &= -4 + u(x-\pi)(-(-4) + 3\sin 2x) \\
&= -4 + u(x-\pi)(4 + 3\sin(2(x-\pi) + 2\pi) \\
&= -4 + u(x-\pi)(4 + 3\sin 2(x-\pi))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\{f(x)\}(s) &= \frac{-4}{s} + \frac{4e^{-\pi s}}{s} + 3e^{-\pi s}\frac{2}{s^2+4} \\
&= \frac{-4}{s} + e^{-\pi s}\left(\frac{4}{s} + \frac{6}{s^2+4}\right)
\end{aligned}
$$

Note. If f is a polynomial of degree n and a is a number then Taylor's Theorem tells us that

$$
f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k
$$

Thus

$$
\mathcal{L}\{u(x-a)f(x)\}(s) = \mathcal{L}\left\{u(x-a)\left(f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right)\right\}(s)
$$

Additional Examples: See Section 4.4 of the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for Section 4.4. The answers are posted on Dr. Walker's web site.