Section 5.3

Before continuing, please read Sections 5.1 and 5.2 in the text for background information. If you need help understanding this material, watch the first part of Dr. Caglar's video 1 for Chapter 5.

Section 5.3
Linear Algebra - Introduction - Solving Systems

Basics

Suppose that each of m and n is a positive integer, A_{ij} is a number for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and B_i is a number for $i = 1, \ldots, m$. We will be concerned with finding the ordered lists of numbers $(x_1, \ldots, x_n)$ (called solutions) such that

$$\begin{array}{ccccccccc} A_{11}x_1 & + & A_{12}x_2 & + & \cdots & + & A_{1n}x_n & = & B_1 \\ A_{21}x_1 & + & A_{22}x_2 & + & \cdots & + & A_{2n}x_n & = & B_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ A_{m1}x_1 & + & A_{m2}x_2 & + & \cdots & + & A_{mn}x_n & = & B_m \end{array}$$

or showing that there is no such list. The numbers A_{ij} are called the coefficients of the system and the x_j are called the unknowns.

If m and n are small we may avoid subscripts and use distinct letters. In case $m = n = 2$ we have

$$ax + by = e$$
$$cx + dy = f$$

Assuming that at least one of a and b is not zero and at least one of c and d is not zero, each equation is that of a line in the plane. One of three things happens: The lines intersect at one point and the system has a unique solution. The lines are parallel so do not intersect and the system has no solution. The two equations describe the same line and every point on that line (there are infinitely many of them) is a solution. If $m = n = 3$ and at least one coefficient in each equation is nonzero, each equation is that of a plane. Again one of three things happens.

There is a unique solution.

There is no solution.

There are infinitely many solutions.

As we shall see, this is also true for the more general system (1).

In order to solve the system (1) we will pass through equivalent systems coming to one where the solution(s) or lack thereof can easily be seen or calculated. Equivalent systems are those that have the same solutions.

Here are three important ways to pass from a system to an equivalent one.

Interchange two equations.

Multiply each side of some equation by a nonzero number.

Replace an equation by itself plus a multiple on some other equation.

These are denoted as follows.

$$E_i \leftrightarrow E_j$$
$$cE_i \to E_i$$
$$E_j + cE_i \to E_j$$

The first procedure that we will use is called Gaussian elimination with backward substitution: Eliminate x_1 from all but the first equation, eliminate x_2 from all but the first two equations, eliminate x_3 from all but the first three equations, et c. Then work from the bottom up.

Solve the system

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3x - 9y + 21z &= 0 \\ x + 6y - 11z &= 1 \end{aligned}$$

Solution. Begin by eliminating x from the second and third equation then eliminate y from the third equation.

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3x - 9y + 21z &= 0 \\ x + 6y - 11z &= 1 \end{aligned}$$

$$\xrightarrow[\longrightarrow]{} 3E_1 + E_2 \to E_2$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3y + 6z &= -3 \\ x + 6y - 11z &= 1 \end{aligned}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3y + 6z &= -3 \\ x + 6y - 11z &= 1 \end{aligned}$$

$$\xrightarrow{\overrightarrow{-E_1 + E_3 \to E_3}}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3y + 6z &= -3 \\ 4y - 6z &= 2 \end{aligned}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3y + 6z &= -3 \\ 4y - 6z &= 2 \end{aligned}$$

$$\xrightarrow{\overrightarrow{-\frac{1}{3}E_2 \to E_2}}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ 4y - 6z &= 2 \end{aligned}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ 4y - 6z &= 2 \end{aligned}$$

$$\xrightarrow{\overrightarrow{-4E_2 + E_3 \to E_3}}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ 2z &= -2 \end{aligned}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ 2z &= -2 \end{aligned}$$

$$\xrightarrow{\frac{1}{2}E_3 \to E_3}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ z &= -1 \end{aligned}$$

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ z &= -1 \end{aligned}$$

Working up, starting with the last equation we have

$$z = -1$$

Then from the second equation

$$y - 2 \cdot (-1) = 1$$

So

$$y = -1.$$

Then from the first equation

$$x + 2 \cdot (-1) - 5 \cdot (-1) = -1$$

So

$$x = -4$$

The given system has a unique solution

$$(x, y, z) = (-4, -1, -1)$$

Example *Solve the system*

$$\begin{aligned} 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

Solution. The elimination process is easier if fractions can be avoided, so we begin by interchanging the first and last equations.

$$\begin{aligned} 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

$$\xrightarrow[\quad]{E_1 \leftrightarrow E_3}$$

$$\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + z &= 1 \\ 3x - 4y - z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + z &= 1 \\ 3x - 4y - z &= 3 \end{aligned}$$

$$\xrightarrow[\quad]{-2E_1 + E_2 \to E_2}$$

$$\xrightarrow[\quad]{-3E_1 + E_3 \to E_3}$$

$$\begin{aligned} x - 2y + 3z &= 2 \\ y - 5z &= -3 \\ 2y - 10z &= -3 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= 2 \\ y - 5z &= -3 \\ 2y - 10z &= -3 \end{aligned}$$

$$\xrightarrow{-2E_2 + E_3 \to E_3}$$

$$\begin{aligned} x - 2y + 3z &= 2 \\ y - 5z &= -3 \\ 0 &= 3 \end{aligned}$$

We have passed through equivalent systems. If there is a solution to the given system then $0 = 3.$ which, of course, is false. Thus the given system has no solution.

Example *Solve the system*

$$\begin{aligned} x + y - 3z &= 1 \\ 2x + y - 4z &= 0 \\ -3x + 2y - z &= 7 \end{aligned}$$

Solution.

$$\begin{aligned} x + y - 3z &= 1 \\ 2x + y - 4z &= 0 \\ -3x + 2y - z &= 7 \end{aligned}$$

$$\xrightarrow{-2E_1 + E_2 \to E_2}$$

$$\xrightarrow{3E_1 + E_3 \to E_3}$$

$$\begin{aligned} x + y - 3z &= 1 \\ -y + 2z &= -2 \\ 5y - 10z &= 10 \end{aligned}$$

$$\begin{aligned} x + y - 3z &= 1 \\ -y + 2z &= -2 \\ 5y - 10z &= 10 \end{aligned}$$

$$\xrightarrow[]{-E_2 \to E_2}$$

$$\begin{aligned} x + y - 3z &= 1 \\ y - 2z &= 2 \\ 5y - 10z &= 10 \end{aligned}$$

$$\begin{aligned} x + y - 3z &= 1 \\ y - 2z &= 2 \\ 5y - 10z &= 10 \end{aligned}$$

$$\xrightarrow[]{-5E_2 + E_3 \to E_3}$$

$$\begin{aligned} x + y - 3z &= 1 \\ y - 2z &= 2 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x + y - 3z &= 1 \\ y - 2z &= 2 \\ 0 &= 0 \end{aligned}$$

To emphasize the fact that z can have any value, we set

$$z = a$$

Then

$$y = 2 + 2a$$

and

$$x = 1 - (2 + 2a) + 3a$$

So

$$x = -1 + a.$$

Thus (x,y,z) is a solution to the given system if and only if

$$(x,y,z) = (-1 + a, 2 + 2a, a)$$

or

$$(x,y,z) = (-1,2,0) + a(1,2,1)$$

for some number a. The solutions are the points in $\mathbb{R}^3$ on the line through $(-1,2,0)$ with directing vector $(1,2,1)$. Of course, there are infinitely many such points.

Matrix Representation

Associated with the system (1) are two important rectangular arrays, The coefficient matrix

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

and the augmented matrix

$$\left(\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & B_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & B_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} & B_m \end{array}\right)$$

The vertical bar separating the last column from the rest of the matrix serves no useful purpose and may be omitted.

In carrying out Gaussian elimination we need only keep up with the augmented matrix at each stage. The elementary operations on equations that we introduced above become elementary operations on rows of the augmented matrix.

$$\begin{array}{lcl} E_i \leftrightarrow E_j & & R_i \leftrightarrow R_j \\ cE_i \to E_i & \text{become} & c\,R_i \to R_i \\ E_j + cE_i \to E_j & & R_j + c\,R_i \to R_j \end{array}$$

Given the system (1), write the augmented matrix and perform elementary row operations in order to put that matrix in row-echelon form.

Saying that a matrix is in row-echelon form means that every zero row is below every nonzero row (A nonzero row is one with at least one nonzero entry.), the first nonzero entry in each nonzero row is a one, and if row i and row $i+1$ are nonzero rows, the leading one in row $i+1$ is to the right of the leading one in row i. It is a consequence of these conditions that every entry below a leading one is zero. Once the augmented matrix is in row-echelon form, write the corresponding system of equations and work from the bottom up.

Example *Solve the system*

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3x - 9y + 21z &= 0 \\ x + 6y - 11z &= 1 \end{aligned}$$

Solution. For

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3x - 9y + 21z &= 0 \\ x + 6y - 11z &= 1 \end{aligned}$$

the augmented matrix is

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ -3 & -9 & 21 & 0 \\ 1 & 6 & -11 & 1 \end{pmatrix}.$$

Transforming into row-echelon form we have

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ -3 & -9 & 21 & 0 \\ 1 & 6 & -11 & 1 \end{pmatrix} \xrightarrow{3R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & -3 & 6 & -3 \\ 1 & 6 & -11 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & -3 & 6 & -3 \\ 1 & 6 & -11 & 1 \end{pmatrix} \xrightarrow{-R_1 + R_3 \to R_3} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & -3 & 6 & -3 \\ 0 & 4 & -6 & 2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 4 & -6 & 2 \end{pmatrix} \xrightarrow{-4R_2 + R_3 \to R_3} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The corresponding system is

$$\begin{aligned} x + 2y - 5z &= -1 \\ y - 2z &= 1 \\ z &= -1 \end{aligned}$$

which can be solved from the bottom up to yield the fact that there is a unique solution:

$$(x, y, z) = (-4, -1, -1)$$

Example. Solve the system

$$\begin{aligned} 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{pmatrix} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{pmatrix}.$$

Transforming into row-echelon form we have

$$\begin{pmatrix} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 2 & -3 & 1 & 1 \\ 3 & -4 & -1 & 3 \end{pmatrix} \xrightarrow[-2R_1 + R_2 \to R_2]{}$$

$$\begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 3 & -4 & -1 & 3 \end{pmatrix} \xrightarrow[-3R_1 + R_3 \to R_3]{} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 2 & -10 & -3 \end{pmatrix}$$

$$\xrightarrow[-2R_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow[\frac{1}{3}R_3 \to R_3]{} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding system is

$$\begin{aligned} x - 2y + 3z &= 2 \\ y - 5z &= -3 \\ 0 &= 1 \end{aligned}$$

If the given system has a solution then $0 = 1$; so the given system has no solution.

Example. Solve the system

$$\begin{aligned} x + y - 3z &= 1 \\ 2x + y - 4z &= 0 \\ -3x + 2y - z &= 7 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 1 & -4 & 0 \\ -3 & 2 & -1 & 7 \end{pmatrix}.$$

Transforming into row-echelon form we have

$$\begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 1 & -4 & 0 \\ -3 & 2 & -1 & 7 \end{pmatrix} \xrightarrow[-2R_1 + R_2 \to R_2]{} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & -1 & 2 & -2 \\ -3 & 2 & -1 & 7 \end{pmatrix}$$

$$\xrightarrow[3R_1 + R_3 \to R_3]{} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & -1 & 2 & -2 \\ 0 & 5 & -10 & 10 \end{pmatrix} \xrightarrow[-R_2 \to R_2]{} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -10 & 10 \end{pmatrix}$$

$$\xrightarrow[-5R_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding system is

$$\begin{aligned} x + y - 3z &= 1 \\ y - 2z &= 2 \\ 0 &= 0. \end{aligned}$$

As in Example 3, it follows that (x,y,z) is a solution to the given system if and only if

$$(x,y,z) = (-1 + a, 2 + 2a, a)$$

or

$$(x,y,z) = (-1,2,0) + a(1,2,1)$$

for some number a.

Definition. The rank of a matrix is the number of nonzero rows when the matrix has been put into row-echelon form.

Definition. Saying that the system (1) is consistent means that it has a unique solution or it has infinitely many solutions.

Definition. Saying that the system (1) is inconsistent means that it has no solution.

Theorem *If the rank of the augmented matrix for a system of type (1) is greater than the rank of the coefficient matrix, the system has no solution. It is inconsistent.*

This is the case because there will be an implied equation of the form $0 = 1$ as in Example 5.

Theorem *If the rank of the augmented matrix for a system of type (1) is the same as the rank of the coefficient matrix, the system has a unique solution or has infinitely many solutions.*

Remark. In order to have a consistent procedure for expressing the solutions in the case of infinitely many solutions, put he augmented matrix in row-echelon form. The columns 1 through n where the leading ones of the nonzero rows occur are called pivot columns (You won't find this term in the text) . The corresponding unknowns are called the pivot unknowns. The other unknowns are called nonpivot or free unknowns. Assign letters ($a, b, c, \ldots$) in this order to the free unknowns and express the pivot unknowns in terms of them. In Online Quiz 9, s and t are used in place of a and b in some of the questions.

Example. Solve the system

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 &= 7 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 &= 7 \end{aligned}$$

Solution. Starting with the augmented matrix and reducing to roe echelon form, we have

$$\begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{pmatrix} \xrightarrow[-3R_1 + R_2 \to R_2]{} \begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{pmatrix}$$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{} \begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 3 & 6 & -9 & 3 \end{pmatrix}$$

$$\xrightarrow[-3R_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The last matrix is in row-echelon form.

The corresponding system is

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \\ 0 &= 0 \end{aligned}$$

Looking at the row-echelon form, we see that the pivot column are columns 1 and 3 and the nonpivot columns are columns 2, 4, and 5. Thus we set

$$\begin{aligned} x_2 &= a \\ x_4 &= b \text{ and} \\ x_5 &= c \end{aligned}$$

From the second equation in the last system we have

$$x_3 = 1 - 2b + 3c.$$

From the first equation in the last system we have

$$x_1 = 2 + 3a - 2(1 - 2b + 3c) + b - 2c$$

so

$$x_1 = 3a + 5b - 8c$$

Thus (x_1,x_2,x_3,x_4,x_5) is a solution to the given system if and only if

$$\begin{aligned} (x_1,x_2,x_3,x_4,x_5) &= (3a + 5b - 8c, a, 1 - 2b + 3c, b, c) \text{ or} \\ (x_1,x_2,x_3,x_4,x_5) &= (0,0,1,0,0) \\ &\quad + a(3,1,0,0,0) \\ &\quad + b(5,0,-2,1,0) \\ &\quad + c(-8,0,3,0,1) \end{aligned}$$

for some triple of numbers (a,b,c).

Example. Fine the solution to the initial value problem

$$\begin{aligned} y''' - 3y'' - y' + 3y &= 0 \\ y(0) &= -4 \\ y'(0) &= 1 \\ y''(0) &= -1. \end{aligned}$$

Solution. The associated polynomial P is given by

$$P(s) = s^3 - 3s^2 - s + 3.$$

The sum of the coefficients is zero so 1 is a zero of P. Dividing $(s-1)$ into $P(s)$ and factoring the quadratic quotient, we find that

$$P(s) = (s-1)(s+1)(s-3).$$

Thus

$$\begin{aligned} y &= c_1e^x + c_2e^{-x} + c_3e^{3x} \\ y' &= c_1e^x - c_2e^{-x} + 3c_3e^{3x} \\ y'' &= c_1e^x + c_2e^{-x} + 9c_3e^{3x} \\ y''' &= c_1e^x - c_2e^{-x} + 27c_3e^{3x}. \end{aligned}$$

Using the initial conditions we have

$$\begin{aligned} c_1 + c_2 + c_3 &= -4 \\ c_1 - c_2 + 3c_3 &= 1 \\ c_1 + c_2 + 9c_3 &= -1. \end{aligned}$$

The augmented matrix for this system is

$$\begin{pmatrix} 1 & 1 & 1 & -4 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 9 & -1 \end{pmatrix}.$$

Reducing to row-echelon form, we have

$$\begin{pmatrix} 1 & 1 & 1 & -4 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 9 & -1 \end{pmatrix} \xrightarrow[-R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 1 & 1 & -4 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 8 & 3 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{8}R_3 \to R_3]{-\frac{1}{2}R_2 \to R_2} \begin{pmatrix} 1 & 1 & 1 & -4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{8} \end{pmatrix}.$$

Thus

$$\begin{aligned} c_3 &= \frac{3}{8} \\ c_2 &= c_3 - \frac{5}{2} = -\frac{17}{8} \end{aligned}$$

and

$$c_1 = -c_2 - c_3 - 4 = \frac{17}{8} - \frac{3}{8} - 4 = -\frac{9}{4}$$

The solution to the initial value problem is

$$y = -\frac{9}{4}e^{x} - \frac{17}{8}e^{-x} + \frac{3}{8}e^{3x}.$$

Example. Find the value(s) of k such that the following system does not have a unique solution.

$$\begin{aligned} -5x - 4y - 2z &= -12 \\ 4x + 3y + (2+k)z &= -10 \\ x + y + z &= 2 \end{aligned}$$

Solution. The augmented matrix for the system is

$$\begin{pmatrix} -5 & -4 & -2 & -12 \\ 4 & 3 & 2+k & -10 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Reducing this to row-echelon form, we have

$$\begin{pmatrix} -5 & -4 & -2 & -12 \\ 4 & 3 & 2+k & -10 \\ 1 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2+k & -10 \\ -5 & -4 & -2 & -12 \end{pmatrix}$$

$$\xrightarrow[\substack{-4R_1 + R_2 \to R_2 \\ 5R_1 + R_3 \to R_3}]{} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & k-2 & -18 \\ 0 & 1 & 3 & -2 \end{pmatrix}$$

$$\xrightarrow[-R_2 \to R_2]{} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2-k & 18 \\ 0 & 1 & 3 & -2 \end{pmatrix}$$

$$\xrightarrow[-R_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2-k & 18 \\ 0 & 0 & 1+k & -20 \end{pmatrix}$$

From this we see that there will be a unique solution unless $k = -1$. If $k = -1$ there will be the implied contradiction $0 = -20$ meaning that there is no solution. The given system does not have a unique solution if and only if $k = -1$. The case of infinitely many solutions does not occur for any value of k.

Additional Examples. See Section 5.3 of the text and the examples for this section posted on Dr. Walker's web site.

Suggested Problems. Do the odd numbered problems for Section 5.3. The answers are posted on Dr. Walker's web site.