Sections 5.4

Section 5.4
Reduced Row-Echelon Form - Homogeneous Systems

Definition. Saying that a matrix is in reduced row-echelon form means that it is in row-echelon form and if a column contains the leading one of some nonzero row, then all other entries in this column are zero.

Definition. Saying that a matrix B is row-equivalent to a matrix A means that one can be obtained from the other by a finite sequence of elementary row operations.

Note. There may be many matrices in row-echelon form that are row-equivalent to a given matrix, but there is only one that is in reduced row-echelon form.

Note. When solving a linear system, the advantage in taking the augmented matrix to reduced row-echelon form rather than just row-echelon form is that the solution(s) or lack thereof can be seen directly and backward substitution is not needed. The disadvantage is that more computation is required for large systems.

Example. Solve the following system by using elementary row operations to transform the augmented matrix into reduced row-echelon form.

$$\begin{aligned} x + 2y - 5z &= -1 \\ -3x - 9y + 21z &= 0 \\ x + 6y - 11z &= 1 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ -3 & -9 & 21 & 0 \\ 1 & 6 & -11 & 1 \end{pmatrix}.$$

Transforming into reduced row-echelon form we have

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ -3 & -9 & 21 & 0 \\ 1 & 6 & -11 & 1 \end{pmatrix} \xrightarrow[-R_1 + R_3 \to R_3]{3R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & -3 & 6 & -3 \\ 0 & 4 & -6 & 2 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{3}R_2 \to R_2]{} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 4 & -6 & 2 \end{pmatrix} \xrightarrow[-4R_2 + R_3 \to R_3]{} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3 \to R_3]{} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[2R_3 + R_2 \to R_2]{} \begin{pmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[5R_3 + R_1 \to R_1]{} \begin{pmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[-2R_2 + R_1 \to R_1]{} \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The corresponding system is

$$\begin{aligned} x &= -4 \\ y &= -1 \\ z &= -1 \end{aligned}$$

so the unique solution is

$$(x,y,z) = (-4,-1,-1)$$

Homogeneous Systems

Definition. Saying that the system (1) is homogeneous means that the right side of each equation is zere so the system becomes

$$\begin{array}{ccccccccc} A_{11}x_1 & + & A_{12}x_2 & + & \cdots & + & A_{1n}x_n & = & 0 \\ A_{21}x_1 & + & A_{22}x_2 & + & \cdots & + & A_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ A_{m1}x_1 & + & A_{m2}x_2 & + & \cdots & + & A_{mn}x_n & = & 0 \end{array}$$

Note. $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots 0)$ is always a solution when the system is homogeneous. It is called the trivial solution. That does not mean that it is an unimportant solution. A homogeneous system has only the trivial solution or it has infinitely many solutions. The case of no solution does not occur for a homogeneous system.

Theorem. If the number of unknowns in a honogeneous system is more than the number of equations, the systme has a nontriial solution.

Example. The system

$$2x - 2y + 6z = 0$$
$$4x + 2y - 8z = 0$$

has two equations and three unknowns so has a nontrivial solution.

Theorem. Suppose that the rank of the augmented matix is k for a system of m equations in n unknowns. If $k = n$ the only solution is the trivial solution, $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots 0)$. If $k < n$ there are infinitely many solutions.

Additional Examples. See Section 5.4 of the text and those that are posted.

Suggested Problems. Do the odd numbered problems for Section 5.4. The answers are posted on Dr. Walker's web site.