Section 5.5

Matrix Operations

Definition *A matrix is a rectangular array. Saying that A is an $m \times n$ matrix means that A has m rows and n columns. The numbers m and n are called the dimensions of A. When A is a matrix, A_{ij} denotes the entry in the ith row and jth column. Some sources including your text and online material luse lower case letters for the entries. Unless stated otherwise, all matrix entries will be numbers or numerically valued functions.*

Definition *When each of A and B is an $m \times n$ matrix, $A+B$ is the $n \times m$ matrix such that*

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

and $A - B$ is the $m \times n$ matrix such that

$$(A-B)_{ij} = A_{ij} - B_{ij}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. These operations are not defined unless the dimensions of A and the dimensions of B are the same.

Example

$$\begin{pmatrix} 3 & 6 & -8 & 3 \\ 12 & -3 & 3 & 7 \\ 0 & -7 & 5 & 16 \end{pmatrix} + \begin{pmatrix} 5 & 8 & -6 & 4 \\ 3 & 14 & -6 & 5 \\ 13 & 12 & 5 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 14 & -14 & 7 \\ 15 & 11 & -3 & 12 \\ 13 & 5 & 10 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & -8 & 3 \\ 12 & -3 & 3 & 7 \\ 0 & -7 & 5 & 16 \end{pmatrix} - \begin{pmatrix} 5 & 8 & -6 & 4 \\ 3 & 14 & -6 & 5 \\ 13 & 12 & 5 & -3 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -2 & -1 \\ 9 & -17 & 9 & 2 \\ -13 & -19 & 0 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & -8 & 3 \\ 12 & -3 & 3 & 7 \\ 0 & -7 & 5 & 16 \end{pmatrix} + \begin{pmatrix} 5 & 8 & -6 & 4 \\ 3 & 14 & -6 & 5 \end{pmatrix}$$ *is not defined.*

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Definition *When A is an $m \times n$ and c is a number, cA is the $m \times n$ matrix such that*

$$(cA)_{ij} = cA_{ij}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example

$$3\begin{pmatrix} 5 & 8 & -6 & 4 \\ 3 & 14 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 24 & -18 & 12 \\ 9 & 42 & -18 & 15 \end{pmatrix}$$

Definition *A row vector is a matrix with only one row and a column vector is a matrix with only one column*

Example

$(2,-5,7)$ *is a row vector*

$$\begin{pmatrix} 3 \\ 8 \\ -12 \end{pmatrix} \text{ is a column vector}$$

Definition *The product of a p-dimensional row vector (on the left) and a p-dimensional column vector (on the right) is defined as follows.*

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_p \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{pmatrix} = A_1B_1 + A_2B_2 + \cdots + A_pB_p$$

Example

$$(2,-5,7)\begin{pmatrix} 3 \\ 8 \\ -12 \end{pmatrix} = 2 \cdot 3 + (-5) \cdot 8 + 7 \cdot (-12) = -118$$

Definition *When A is an $m \times p$ matrix and B is a $p \times n$ matrix, the matrix product AB is the $m \times n$ matrix whose i-th row and j-th column entry is the i-th row of A times the j-th column of B. The matrix product AB is not defined unless the number of columns in A is the same as the number of rows in B.*

Example

$$\begin{pmatrix} 2 & -3 & 2 \\ 4 & -5 & 3 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 3 \\ 5 & 7 & 2 \\ 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} -15 & -15 & 10 \\ -26 & -26 & 17 \\ -7 & -2 & 19 \end{pmatrix}$$

Remark *Even when each of A and B is $n \times n$ so that each of AB and BA is defined, it is usually the case that*

$$AB \neq BA.$$

Matrix multiplication is not commutative.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

while

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

Remark *Whenever the operations are defined*

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

Special Matrices

Definition *The $m \times n$ zero matrix, $0_{m \times n}$ or 0 if the dimension is clear, is the $m \times n$ matrix with every entry 0. The $n \times n$ identity matrix, I_n or I if the dimension is clear, is the $n \times n$ matrix whose i-row and j-th column entry is 1 when $i = j$ and is 0 when $i \neq j$.*

Remark *$A + 0 = A$, $0 + A = A$, $AI = A$, and $IA = A$ whenever the indicated operations are defined.*

Matrix Formulation

Remark *The system (1) in section 5.3 can be expressed*

$$AX = B$$

where A is the $m \times n$ coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}.$$

Problems

Do the odd numbered problems 1-15 of Sections 5.5.