Sections 5.7

Section 5.7
Vector Spaces

Definition. A vector space consists of a set whose members are called vectors, a field (the real numbers or the complex numbers in this course) whose members are called scalars, and two operations where the following conditions are satisfied.

- The first operation is called addition of vectors. When each of A and B is a vector, there is a vector denoted $A + B$.
- The second operation is called multiplication of scalars and vectors. When c is a scalar and A is a vector, there is a vector denoted cA.
- Addition of vectors is commutative and associative. When each of A, B, and C is a vector
$$A + B = B + A$$
and
$$A + (B + C) = (A + B) + C.$$
- There is a unique vector called the zero vector and denoted $\mathbf{0}$ with the property that if A is a vector then
$$A + \mathbf{0} = A.$$
- If A is a vector, there is a unique vector B such that $A + B = \mathbf{0}$. (This vector B is denoted $-A$)
- If c is a scalar and each of A and B is a vector, then
$$c(A + B) = cA + cB.$$
- If each of c and d is a scalar and A is a vector, then
$$(c + d)A = cA + dA \text{ and } a(cA) = (ac)A.$$
- If A is a vector, then the scalar 1 times the vector A is A.

When the field of scalars is the real number system, the vector space is called a real vector space.

When the field of scalars is the complex number system, the vector space is called a complex vector space.

Theorem. In a vector space, if A is a vector and c is a scalar, then

$$0A = \mathbf{0},$$

$$(-1)A = -A,$$

and

$$c\mathbf{0} = \mathbf{0}.$$

Definition. In a vector space,

$$A - B = A + (-B).$$

Examples of Real Vector Spaces.

- $\mathbb{R}^n$- all n-tuples or n-dimensional row vectors of real numbers.

$$(x_1,\ldots,x_n) + (y_1,\ldots y_n) = (x_1 + y_1,\ldots,x_n + y_n),$$

$$c(x_1,\ldots,x_n) = (cx_1,\ldots,cx_n),$$

$$\mathbf{0} = (0,\ldots,0),$$

and

$$-(x_1,\ldots,x_n) = (-x_1,\ldots,-x_n).$$

- $\mathbb{R}^{m\times n}$- all $m \times n$ matrices of real numbers with the usual addition of matrices and multiplication of scalars and matrices. The zero vector is the $m \times n$ zero matrix. The additive inverse of a matrix is obtained by taking the negative of each of its entries.

  Aspecial case of this is $\mathbb{R}^{m\times 1}$, the space of all m-dimensional column vectors.

- When S is a set, the collection of all real valued functions having domain the set S.

$$(f+g)(x) = f(x) + g(x) \text{ and } (cf)(x) = cf(x) \text{ for all } x \text{ in } S,$$

$$\mathbf{0}(x) = 0 \text{ for all } x \text{ in } S,$$

and

$$(-f)(x) = -f(x) \text{ for all } x \text{ in } S.$$

- When V is a real vector space and S is a set, the collection of all V valued functions having domain the set S.

Examples of Complex Vector Spaces.

- $\mathbb{C}^n$- all n-tuples or n-dimensional row vectors of complex numbers.

$$(x_1,\ldots,x_n) + (y_1,\ldots y_n) = (x_1 + y_1,\ldots,x_n + y_n),$$

$$c(x_1,\ldots,x_n) = (cx_1,\ldots,cx_n),$$

$$\mathbf{0} = (0, \dots, 0),$$

and

$$-(x_1, \dots, x_n) = (-x_1, \dots, -x_n).$$

- $\mathbb{C}^{m \times n}$- all $m \times n$ matrices of complex numbers with the usual addition of matrices and multiplication of scalars and matrices. The zero vector is the $m \times n$ zero matrix. The additive inverse of a matrix is obtained by taking the negative of each of its entries.

- When S is a set, the collection of all complex valued functions having domain the set S.

$$(f+g)(x) = f(x) + g(x) \text{ and } (cf)(x) = cf(x) \text{ for all } x \text{ in } S,$$

$$\mathbf{0}(x) = 0 \text{ for all } x \text{ in } S,$$

and

$$(-f)(x) = -f(x) \text{ for all } x \text{ in } S.$$

- When V is a complex vector space and S is a set, the collection of all V valued functions having domain the set S.

Magnitude **and Direction**

In $\mathbb{R}^2$, the magnitude and direction associated with vector (x_1,x_2) is that of the directed line segment from the origin to (x_1,x_2). At each point (a,b), the representative of (x_1,x_2) at (a,b) is the directed line segment from (a,b) to $(a+x_1,b+x_2)$.

In $\mathbb{R}^3$, the magnitude and direction associated with vector (x_1,x_2,x_3) is that of the directed line segment from the origin to (x_1,x_2,x_3). At each point (a,b,c), the representative of (x_1,x_2,x_3) at (a,b,c) is the directed line segment from (a,b,c) to $(a+x_1,b+x_2,c+x_3)$.

There is no notion of magnitude and direction in vector spaces of functions.

Definition. Saying that a list of vectors $(v_1, \dots, v_m)$ in a vector space is linearly independent means that if $(c_1, \dots, c_m)$ is a list of scalars and

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}$$

then

$$c_1 = \cdots = c_m = 0$$

Saying that the list of vectors is linearly dependent means that it is not linearly independent.

Note. A list of vectors $(v_1, \dots, v_m)$ is linearly dependent if and only it there is a list of scalars $(c_1, \dots, c_m)$ at lease one of which is not zero such that

$$c_1v_1 + \cdots + c_mv_m = \mathbf{0}.$$

Theorem. A list of vectors $(v_1, \ldots, v_m)$ is linearly dependent if and only if one of the vectors is a linear combination of the others.

$$v_k = c_1v_1 + \cdots + c_{k-1}v_{k-1} + c_{k+1}v_{k+1} + \cdots c_mv_m.$$

As a special case of this we have that a pair of vectors (v_1, v_2) is linearly dependent if and only if

$$v_1 = cv_2$$

for some scalar c or

$$v_2 = dv_1$$

for some scalar d.

Tests for Independence and Dependence.

Theorem. Suppose that $(v_1, \ldots, v_m)$ is a list of vectors in $\mathbb{R}^n$. If $m > n$ (more vectors than the dimension of the space) then $(v_1, \ldots, v_m)$ is linearly dependent.

If $m = n$ let A be the $n \times n$ matrix whose i-th row v_i for $i = 1, \ldots, n$. If $\det A \neq 0$ then $(v_1, \ldots, v_m)$ is linearly independent. If $\det A = 0$ then $(v_1, \ldots, v_m)$ is linearly dependent.

If $m < n$, solve the system

$$c_1v_1 + \cdots + c_mv_m = \mathbf{0}$$

for $(c_1, \ldots, c_m)$. If the only solution is $c_1 = c_2 = \cdots = c_m = 0$, then $(v_1, \ldots, v_m)$ is linearly independent. If there is a solution where at least one $c_k \neq 0$, then $(v_1, \ldots, v_m)$ is linearly dependent.

Multiplying a matrix on the right by a column vector of the correct dimension produces a linear combination of the columns of the matrix.

Example.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \begin{pmatrix} a \\ d \\ g \end{pmatrix} + c_2 \begin{pmatrix} b \\ e \\ h \end{pmatrix} + c_3 \begin{pmatrix} c \\ f \\ i \end{pmatrix}$$

Note. The system

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}$$

is equivalent to

$$c_1 v_1^T + \cdots + c_m v_m^T = \mathbf{0}^T$$

which is equivalent to

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

where A is the $n \times m$ matrix whose j-th column is v_j^T for $j = 1, \ldots, m$.

Example. Let $v_1 = (1, -2, 3)$, $v_2 = (-2, 4, 1)$, and $v_3 = (-4, 8, 9)$. Show whether the list (v_1, v_2, v_3) of three vectors in $\mathbb{R}^3$ is linearly independent or linearly depepndet.

Solution. Using two type three row operations, $2R_1 + R_2 \to R_2$ and $4R_1 + R_3 \to R3$, we have

$$\det \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & 1 \\ -4 & 8 & 9 \end{pmatrix} = \det \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 21 \end{pmatrix} = 1 \cdot 0 \cdot 21 = 0.$$

So (v_1, v_2, v_3) is linearly dependent.

Example. Let $v_1 = (1, -2, 3)$, $v_2 = (-2, 4, 1)$, and $v_3 = (-4, 8, 9)$. Express one of these vectors as a linear combination of the other two.

Solution.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = (0, 0, 0)$$

is, by transposing each side, equivalent to

$$c_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ 3 & 1 & 9 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system we have

$$\begin{pmatrix} 1 & -2 & -4 & 0 \\ -2 & 4 & 8 & 0 \\ 3 & 1 & 9 & 0 \end{pmatrix} \xrightarrow[2R_1 + R_2 \to R_2]{} \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 9 & 0 \end{pmatrix}$$

$$\xrightarrow[-3R_1 + R_2 \to R_2]{} \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 21 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$c_1 - 2c_2 - 4c_3 = 0$$

$$c_2 + 3c_3 = 0$$

There is one non-pivot or free unknown, c_3. So set

$$c_3 = a.$$

Then

$$c_2 = -3a$$

and

$$c_1 - 2(-3a) - 4a = 0 \text{ so } c_1 = -2a$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = (0,0,0)$$

if and only if

$$(c_{1,} c_2, c_3) = a(-2,-3,1)$$

for some number a. Letting $a = 1$ we see that

$$-2v_1 - 3v_2 + v_3 = (0,0,0).$$

Thus

$$v_3 = 2v_1 + 3v_2.$$

Definition. When $(y_1, \ldots, y_n)$ is a list of functions each defined on an interval J and each having $n-1$ derivatives, their **Wronski matrix** is given by

$$M_W[y_1,\ldots,y_n] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

and their **Wronskian** is given by

$$W[y_1,\ldots,y_n] = \det\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Theorem. (**First Wronskian Test**) If $W[y_1,\ldots,y_n](x_0) \neq 0$ for some number x_0 in the interval J, then $(y_1,\ldots,y_n)$ a list of functions that are linearly independent over J.I

Additional Examples: See Section of 5.7 the text and the notes presented on the board in class.

Suggested Problems. Do the odd numbered problems for section 5.7. The answers are posted on Dr. Walker's web site.