Engineering Mathematics

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Section 5.7
Vector Spaces

Definition. A vector space consists of a set whose members are called vectors, a field (the real numbers or the complex numbers in this course) whose members are called scalars, and two operations where the following conditions are satisfied.

- The first operation is called addition of vectors. When each of A and B is a vector, there is a vector denoted $A + B$.
- The second operation is called multiplication of scalars and vectors. When c is a scalar and A is a vector, there is a vector denoted cA.
- Addition of vectors is commutative and associative. When each of A, B, and C is a vector
$$A + B = B + A$$
and
$$A + (B + C) = (A + B) + C.$$

- There is a unique vector called the zero vector and denoted $\mathbf{0}$ with the property that if A is a vector then

$$A + \mathbf{0} = A.$$

- If A is a vector, there is a unique vector B such that $A + B = \mathbf{0}$. (This vector B is denoted $-A$)
- If c is a scalar and each of A and B is a vector, then

$$c(A + B) = cA + cB.$$

- If each of c and d is a scalar and A is a vector, then

$$(c + d)A = cA + dA \text{ and } a(cA) = (ac)A.$$

- If A is a vector, then the scalar 1 times the vector A is A.

When the field of scalars is the real number system, the vector space is called a real vector space.

When the field of scalars is the complex number system, the vector space is called a complex vector space.

Theorem. In a vector space, if A is a vector and c is a scalar, then

$$0A = \mathbf{0},$$

$$(-1)A = -A,$$

and

$$c\mathbf{0} = \mathbf{0}.$$

Definition. In a vector space,

$$A - B = A + (-B).$$

Examples of Real Vector Spaces.

- $\mathbb{R}^n$- all n-tuples or n-dimensional row vectors of real numbers.

$$(x_1, \ldots, x_n) + (y_1, \ldots y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n),$$

$$\mathbf{0} = (0, \ldots, 0),$$

and

$$-(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n).$$

- $\mathbb{R}^{m\times n}$- all $m \times n$ matrices of real numbers with the usual addition of matrices and multiplication of scalars and matrices. The zero vector is the $m \times n$ zero matrix. The additive inverse of a matrix is obtained by taking the negative of each of its entries.

  Aspecial case of this is $\mathbb{R}^{m\times 1}$, the space of all m-dimensional column vectors.

- When S is a set, the collection of all real valued functions having domain the set S.

$$(f+g)(x) = f(x) + g(x) \text{ and } (cf)(x) = cf(x) \text{ for all } x \text{ in } S,$$

$$\mathbf{0}(x) = 0 \text{ for all } x \text{ in } S,$$

and

$$(-f)(x) = -f(x) \text{ for all } x \text{ in } S.$$

- When V is a real vector space and S is a set, the collection of all V valued functions having domain the set S.

Examples of Complex Vector Spaces.

- $\mathbb{C}^n$- all n-tuples or n-dimensional row vectors of complex numbers.

$$(x_1, \ldots, x_n) + (y_1, \ldots y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n),$$

$$\mathbf{0} = (0, \ldots, 0),$$

  and

$$-(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n).$$

- $\mathbb{C}^{m\times n}$- all $m \times n$ matrices of complex numbers with the usual addition of matrices and multiplication of scalars and matrices. The zero vector is the $m \times n$ zero matrix. The additive inverse of a matrix is obtained by taking the negative of each of its entries.

- When S is a set, the collection of all complex valued functions having domain the set S.

$$(f+g)(x) = f(x) + g(x) \text{ and } (cf)(x) = cf(x) \text{ for all } x \text{ in } S,$$

$$\mathbf{0}(x) = 0 \text{ for all } x \text{ in } S,$$

and

$$(-f)(x) = -f(x) \text{ for all } x \text{ in } S.$$

- When V is a complex vector space and S is a set, the collection of all V valued functions having domain the set S.

Magnitude **and Direction**

In $\mathbb{R}^2$, the magnitude and direction associated with vector (x_1, x_2) is that of the directed line segment from the origin to (x_1, x_2). At each point (a, b), the representative of (x_1, x_2) at (a, b) is the directed line segment from (a, b) to $(a + x_1, b + x_2)$.

In $\mathbb{R}^3$, the magnitude and direction associated with vector (x_1, x_2, x_3) is that of the directed line segment from the origin to (x_1, x_2, x_3). At each point (a, b, c), the representative of (x_1, x_2, x_3) at (a, b, c) is the directed line segment from (a, b, c) to $(a + x_1, b + x_2, c + x_3)$.

There is no notion of magnitude and direction in vector spaces of functions.

Definition. Saying that a list of vectors $(v_1, \ldots, v_m)$ in a vector space is linearly independent means that if $(c_1, \ldots, c_m)$ is a list of scalars and

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}$$

then

$$c_1 = \cdots = c_m = 0$$

Saying that the list of vectors is linearly dependent means that it is not linearly independent.

Note. A list of vectors $(v_1, \ldots, v_m)$ is linearly dependent if and only it there is a list of scalars $(c_1, \ldots, c_m)$ at lease one of which is not zero such that

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}.$$

Theorem. A list of vectors $(v_1, \ldots, v_m)$ is linearly dependent if and only if one of the vectors is a linear combination of the others.

$$v_k = c_1 v_1 + \cdots + c_{k-1} v_{k-1} + c_{k+1} v_{k+1} + \cdots c_m v_m .$$

As a special case of this we have that a pair of vectors (v_1, v_2) is linearly dependent if and only if

$$v_1 = cv_2$$

for some scalar c or

$$v_2 = dv_1$$

for some scalar d.

Tests for Independence and Dependence.

Theorem. Suppose that $(v_1, \ldots, v_m)$ is a list of vectors in $\mathbb{R}^n$. If $m > n$ (more vectors than the dimension of the space) then $(v_1, \ldots, v_m)$ is linearly dependent.

If $m = n$ let A be the $n \times n$ matrix whose i-th row v_i for $i = 1, \ldots, n$. If $\det A \neq 0$ then $(v_1, \ldots, v_m)$ is linearly independent. If $\det A = 0$ then $(v_1, \ldots, v_m)$ is linearly dependent.

If $m < n$, solve the system

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}$$

for $(c_1, \ldots, c_m)$. If the only solution is $c_1 = c_2 = \cdots = c_m = 0$, then $(v_1, \ldots, v_m)$ is linearly independent. If there is a solution where at least one $c_k \neq 0$, then $(v_1, \ldots, v_m)$ is linearly dependent.

Multiplying a matrix on the right by a column vector of the correct dimension produces a linear combination of the columns of the matrix.

Example.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \begin{pmatrix} a \\ d \\ g \end{pmatrix} + c_2 \begin{pmatrix} b \\ e \\ h \end{pmatrix} + c_3 \begin{pmatrix} c \\ f \\ i \end{pmatrix}$$

Note. The system

$$c_1 v_1 + \cdots + c_m v_m = \mathbf{0}$$

is equivalent to

$$c_1 v_1^T + \cdots + c_m v_m^T = \mathbf{0}^T$$

which is equivalent to

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

where A is the $n \times m$ matrix whose j-th column is v_j^T for $j = 1, \ldots, m$.

Example. Let $v_1 = (1, -2, 3)$, $v_2 = (-2, 4, 1)$, and $v_3 = (-4, 8, 9)$. Show whether the list (v_1, v_2, v_3) of three vectors in $\mathbb{R}^3$ is linearly independent or linearly depepndet.

Solution. Using two type three row operations, $2R_1 + R_2 \longrightarrow R_2$ and $4R_1 + R_3 \longrightarrow R3$, we have

$$\det \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & 1 \\ -4 & 8 & 9 \end{pmatrix} = \det \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 21 \end{pmatrix} = 1 \cdot 0 \cdot 21 = 0.$$

So (v_1, v_2, v_3) is linearly dependent.

Example. Let $v_1 = (1, -2, 3)$, $v_2 = (-2, 4, 1)$, and $v_3 = (-4, 8, 9)$. Express one of these vectors as a linear combination of the other two.

Solution.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = (0, 0, 0)$$

is, by transposing each side, equivalent to

$$c_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ 3 & 1 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system we have

$$\left(\begin{array}{cccc} 1 & -2 & -4 & 0 \\ -2 & 4 & 8 & 0 \\ 3 & 1 & 9 & 0 \end{array}\right) \xrightarrow[2R_1 + R_2 \to R_2]{} \left(\begin{array}{cccc} 1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 9 & 0 \end{array}\right)$$

$$\xrightarrow[-3R_1 + R_2 \to R_2]{} \left(\begin{array}{cccc} 1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 21 & 0 \end{array}\right) \to \left(\begin{array}{cccc} 1 & -2 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$c_1 - 2c_2 - 4c_3 = 0$$

$$c_2 + 3c_3 = 0$$

There is one non-pivot or free unknown, c_3. So set

$$c_3 = a.$$

Then

$$c_2 = -3a$$

and

$$c_1 - 2(-3a) - 4a = 0 \text{ so } c_1 = -2a$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = (0, 0, 0)$$

if and only if

$$(c_{1,} c_2, c_3) = a(-2, -3, 1)$$

for some number a. Letting $a = 1$ we see that

$$-2v_1 - 3v_2 + v_3 = (0, 0, 0).$$

Thus

$$v_3 = 2v_1 + 3v_2.$$

Definition. When $(y_1, \ldots, y_n)$ is a list of functions each defined on an interval J and each having $n-1$ derivatives, their **Wronski matrix** is given by

$$M_W[y_1, \ldots, y_n] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

and their **Wronskian** is given by

$$W[y_1, \ldots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Theorem. (First Wronskian Test) If $W[y_1, \dots, y_n](x_0) \neq 0$ for some number x_0 in the interval J, then $(y_1, \dots, y_n)$ a list of functions that are linearly independent over J.I

Additional Examples: See Section of 5.7 the text.

Suggested Problems. Do the odd numbered problems for section 5.7. The answers are posted on Dr. Walker's web site.