Sections 5.8

Section 5.8
Eigenvalues and Eigenvectors

Definition. A nonzero vector is one with at least one nonzero entry.

Definition. Suppose that A is an $n \times n$ matrix. The statement that λ_0 is an **eigenvalue** for A means that λ_0 is a scalar and

$$AK = \lambda_0 K$$

for some nonzero n-dimensional column vector K.

Definition. Suppose that λ_0 is an eigenvalue for an $n \times n$ matrix A. The statement that K is a corresponding **eignevector** means that K is a nonzero n-dimensional column vector and

$$AK = \lambda_0 K.$$

The **eigenspace** corresponding to λ_0 consists of all n-dimensional column vectors K such that

$$AK = \lambda_0 K.$$

Note that the eigenspace consists of all the corresponding eigenvectors together with the n-dimensional zero column vector.

Note that eigenvector are presented as row vectors in Section 5.8 of the text. You will need to transpose the row vectors to get eigenvectors as we have defined them. For example, if the text presents $(2, -1, 5)$ as an eigenvector, the eigenvector is actually

$$(2, -1, 5)^T = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Definition. The characteristic polynomial for an $n \times n$ matrix A is the function $\mathcal{P}$ given by

$$\mathcal{P}(\lambda) = \det(A - \lambda I_n)$$

for all complex numbers λ.

Theorem. The number λ_0 is an eigenvalue for A if and only if $\mathcal{P}(\lambda_0) = 0$

Proof. $\mathcal{P}(\lambda_0) = 0$ if and only if $A - \lambda_0 I_n$ is not invertible, if and only if $(A - \lambda_0 I_n)K = \mathbf{0}$ has a solution for some nonzero n-dimensional column vector K, if and only if $AK = \lambda_0 K$ for some nonzero n-dimensional column vector K, if and only if $\lambda_0$is an eigenvalue for A.

Note. Eigenvectors are not unique. if K is an eigenvector for A corresponding to the eigenvalue $\lambda_0$and c is a nonzero scalar, then cK is also an eigenvector for A corresponding to λ_0.

If $AK = \lambda_0 K$ then $c(AK) = c(\lambda_0 K)$ so $A(cK) = \lambda_0(cK)$.

Theorem. Suppose that λ_0 is an eigenvalue for the $n \times n$ matrix A.

$$AK = \lambda_0 K \text{ if and only if } (A - \lambda_0 I_n)K = \mathbf{0}.$$

So K is in the eigenspace corresponding to $\lambda_0$if and only if K is an n-dimensional column vector satisfying $(A - \lambda_0 I_n)K = \mathbf{0}$,

and K is an eigenvector corresponding to λ_0 if and only if K is a nonzero n-dimensional column vector satisfying $(A - \lambda_0 I_n)K = \mathbf{0}$.

Note. If λ_0 is an eigenvalue for the matrix A, to find the corresponding eigenvectors, find all the solutions K to

$$(A - \lambda_0 I_n)K = \mathbf{0}$$

then discard the n-dimensional zero column vector.

Definition. When λ_0 is an eigenvalue for the $n \times n$ matrix A, the **dimension of the corresponding eigenspace** is the maximum number of vectors possible in an independent list of corresponding eigenvectors.

Theorem. When λ_0 is an eigenvalue for the $n \times n$ matrix A, the dimension of the corresponding eigenspace is the number of all zero rows when the matrix $A - \lambda_0 I$ has been put into row echelon form.

Note. Eigenvectors corresponding to different eigenvalues are independent, so the maximum number of independent eigenvectors for a matrix is the sum of the dimensions of its eigenspaces.

Example. Let

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}.$$

The characteristic polynomial is $\mathcal{P}$ where

$$\mathcal{P}(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{pmatrix}$$

$$= (-1-\lambda)(8-\lambda) + 14 = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6)$$

so the eigenvalues are 1 and 6.

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is in the eigenspace corresponding to the eigenvalue 1 if and only if

$$(A - (1)I)K = \mathbf{0}$$

or $$\left(\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix} - (1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\begin{pmatrix} -2 & 2 \\ -7 & 7 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augmented matrix for this system is

$$\begin{pmatrix} -2 & 2 & 0 \\ -7 & 7 & 0 \end{pmatrix}$$

which has

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as its row echelon form. Note that there is exactly one all zero row so the eigenspace is one dimensional. The implied equation is

$$k_1 - k_2 = 0.$$

k_2 is free so letting $k_2 = a$ we have $k_1 = a$ and see that K is in the eigenspace if and only if

$$K = \begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for some number } a.$$

Consequently, K is an eigenvector corresponding to the eigenvalue 1 if and only if

$$K = \begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for some number } a \neq 0.$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is one eigenvector.}$$

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is in the eigenspace corresponding to the eigenvalue 6 if and only if

$$(A - 6I)K = \mathbf{0}$$

$$\text{or } \left(\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} -7 & 2 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augmented matrix for this system is

$$\begin{pmatrix} -7 & 2 & 0 \\ -7 & 2 & 0 \end{pmatrix}$$

which has

$$\begin{pmatrix} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as its row echelon form. Note that there is exactly one all zero row so the eigenspace is one dimensional. The implied equation is

$$k_1 - \frac{2}{7}k_2 = 0.$$

k_2 is free so letting $k_2 = a$ we have $k_1 = \frac{2}{7}a$ and see that K is in the eigenspace if and only if

$$K = \begin{pmatrix} \frac{2}{7}a \\ a \end{pmatrix} = a\begin{pmatrix} \frac{2}{7} \\ 1 \end{pmatrix} \text{ for some number } a.$$

Consequently, K is an eigenvector corresponding to the eigenvalue 6 if and only if

$$K = \begin{pmatrix} \frac{2}{7}a \\ a \end{pmatrix} = a\begin{pmatrix} \frac{2}{7} \\ 1 \end{pmatrix} \text{ for some number } a \neq 0.$$

Letting $a = 7$ to avoid fractions, we see that

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} \text{ is one eigenvector.}$$

Example. Let

$$A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}.$$

The characteristic polynomial is $\mathcal{P}$ where

$$\begin{aligned} \mathcal{P}(\lambda) &= \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{pmatrix} \\ &= (3-\lambda)(7-\lambda) + 4 = \lambda^2 - 10\lambda + 25 \\ &= (\lambda - 5)^2. \end{aligned}$$

So there is only one eigenvalue. It is the number 5.

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is in the eigenspace corresponding to the eigenvalue 5 if and only if

$$(A - 5I)K = \mathbf{0}$$

$$\text{or} \begin{pmatrix} 3-5 & 4 \\ -1 & 7-5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{or} \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augmented matrix for this system is

$$\begin{pmatrix} -2 & 4 & 0 \\ -1 & 2 & 0 \end{pmatrix}$$

which has

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as its row echelon form. Note that there is exactly one all zero row so the eigenspace is one dimensional. The implied equation is

$$k_1 - 2k_2 = 0.$$

k_2 is free so letting $k_2 = a$ we have $k_1 = 2a$ and see that K is in the eigenspace if and only if

$$K = \begin{pmatrix} 2a \\ a \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for some number } a.$$

Consequently, K is an eigenvector corresponding to the eigenvalue 5 if and only if

$$K = \begin{pmatrix} 2a \\ a \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for some number } a \neq 0.$$

Letting $a = 1$, we see that

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ is one eigenvector.}$$

Note. Suppose that A is a real $n \times n$ matrix that has a non real complex eigenvalue λ and corresponding eigenvector K.

$$AK = \lambda K \text{ so } \overline{AK} = \overline{\lambda K} \text{ so } \overline{A}\overline{K} = \overline{\lambda}\overline{K}$$

$$\overline{A} = A \text{ since } A \text{ is real, so}$$

$$A\overline{K} = \overline{\lambda}\overline{K}$$

It follows that $\overline{\lambda}$ is and eigenvalue and $\overline{K}$ is a corresponding eigenvector

Example. Let

$$A = \begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}.$$

The characteristic polynomial is $\mathcal{P}$ where

$$\mathcal{P}(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{pmatrix} = \lambda^2 + 9$$

So the eigenvalues and $3i$ and $-3i$.

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is in the eigenspace corresponding to the eigenvalue $3i$ if and only if

$$(A - 3iI)K = \mathbf{0}$$

or $\begin{pmatrix} -1-3i & 2 \\ -5 & 1-3i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The augmented matrix for this system is

$$\begin{pmatrix} -1-3i & 2 & 0 \\ -5 & 1-3i & 0 \end{pmatrix} \xrightarrow{\frac{1}{-1-3i}R_1 \to R_1} \begin{pmatrix} 1 & \frac{2}{-1-3i} & 0 \\ -5 & 1-3i & 0 \end{pmatrix}$$

$$\xrightarrow{5R_1 + R_2 \to R_2} \begin{pmatrix} 1 & \frac{2}{-1-3i} & 0 \\ 0 & \frac{10}{-1-3i} + 1 - 3i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{2}{-1-3i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that there is exactly one all zero row so the eigenspace is one dimensional. The implied equation is

$$.k_1 + \frac{2}{-1-3i} k_2 = 0$$

k_2 is free so setting $k_2 = a$ we have $k_1 = \frac{2}{1+3i} a$ and see that K is in the eigenspace if and only if

$$K = \begin{pmatrix} \frac{2}{1+3i} a \\ a \end{pmatrix} = a \begin{pmatrix} \frac{2}{1+3i} \\ 1 \end{pmatrix} \text{ for some number } a.$$

Taking $a = 1 + 3i$ we see that

$$\begin{pmatrix} 2 \\ 1+3i \end{pmatrix}$$

is one eigenvector corresponding to the eigenvalue $3i$. Taking conjugates, we have that

$$\begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue $-3i$.

Additional Examples: See Section of 5.8 the text and the other videos posted for this section.

Suggested Problems. Do the odd numbered problems for section 5.8. The answers are posted on Dr. Walker's web site.