Section 6.1

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Systems of Linear Differential Equations

We will be concerned with systems of the form

$$\begin{aligned}
x_1'(t) &= A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + \cdots + A_{1n}(t)x_n(t) + f_1(t) \\
x_2'(t) &= A_{21}(t)x_1(t) + A_{22}(t)x_2(t) + \cdots + A_{2n}(t)x_n(t) + f_2(t) \\
&\vdots \\
x_n'(t) &= A_{n1}(t)x_1(t) + A_{n2}(t)x_2(t) + \cdots + A_{nn}(t)x_n(t) + f_n(t)
\end{aligned}$$

for all t in an interval J. Using vector-matrix notation this becomes

$$X' = AX + F$$

where A is the coefficient matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \text{ and } F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Saying tht the system is **homogeneous** means that

$$F = \mathbf{0}.$$

Note that

$$X' = AX$$

is equivalent to

$$X' - AX = \mathbf{0}$$

The uniqueness and existence theoem for the homogeneous system as follows.

Theorem. Suppose that J is an interval, t_0 is a number in J, E is an n-dimensional constant column vector, and A is an $n \times n$ matrix of continuous functions defined on J. There is a unique n-dimensional column vector function X such that

$$X' = AX \text{ on } J \text{ and } X(t_0) = E.$$

Every n-th order scalar linear differential equation can be fromulated as a first order system of the type we are considering here.

Example. Suppose that

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$$

for all t in an interval J. Let

$$x_1 = y \text{ and } x_2 = y'.$$

Note that

$$y'' = -py' - qy + f.$$

Thus

$$x_1' = x_2 \text{ and } x_2' = -px_2 - qx_1 + f.$$

$$x_1' = x_2 \text{ and } x_2' = -px_2 - qx_1 + f.$$

or

$$x_1' = 0 \cdot x_1 + 1 \cdot x_2 + 0 \text{ and } x_2' = -qx_1 - px_2 + f.$$

or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Example. Suppose that

$$y''' + p_2y'' + p_1y' + p_0y = f.$$

Let

$$x_1 = y,\ x_2 = y', \text{ and } x_3 = y''.$$

Note that

$$y''' = -p_2y'' - p_1y' - p_0y + f.$$

Thus

$$x_1' = x_2,\ x_2' = x_3, \text{ and } x_3' = -p_2x_3 - p_1x_2 - p_0x_1 + f$$

so

$$X' = AX + F$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}.$$

The standard vector-mtrix formulation of

$$y^{(n)} + p_{n-1}y^{(n-1)} + p_{n-1}y^{(n-2)} + \cdots p_0y = f$$

is

$$X' = AX + F$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{n-1} \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}.$$

Additional Examples. See Section 6.1 of the text and the material posted online.

Suggested Problems. Do the odd numbered problems for Section 6.1.