Section 6.2

Section 6.2
Homogeneous Systems

We will be concerned with systems of the form

$$\begin{aligned} x_1'(t) &= A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + \cdots + A_{1n}(t)x_n(t) \\ x_2'(t) &= A_{21}(t)x_1(t) + A_{22}(t)x_2(t) + \cdots + A_{2n}(t)x_n(t) \\ &\vdots \\ x_n'(t) &= A_{n1}(t)x_1(t) + A_{n2}(t)x_2(t) + \cdots + A_{nn}(t)x_n(t) \end{aligned}$$

for all t in an interval J. Using vector-matrix notation this becomes

$$X' = AX \text{ or } X' - AX = \mathbf{0}$$

where A is the coefficient matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}..$$

The uniqueness and existence theorem for the homogeneous system is as follows.

Theorem. Suppose that J is an interval, t_0 is a number in J, E is an n-dimensional constant column vector, and A is an $n \times n$ matrix of continuous functions defined on J. There is a unique n-dimensional column vector function X such that

$$X' = AX \text{ on } J \text{ and } X(t_0) = E.$$

Theorem. Every linear combination of solutions to

$$X' = AX$$

is also a solution.

Proof. If $X_i' = AX_i$ for $i = 1, \ldots, m$ then
$(c_1X_1 + c_2X_2 + \cdots + c_mX_m)' = c_1X_1' + c_2X_2' + \cdots c_mX_m' = c_1AX_1 + c_2AX_2 + \cdots + c_mAX_m = A(c_1X_1 + c$

Definition. Suppose that X_j is an n-dimensional column vector valued function defined on an interval j for $j = 1, \ldots, m$. Saying that the list of functions $(X_1, X_2, \ldots, X_m)$ is linearly independent over J means that if each of $c_1, c_2, \ldots, c_m$ is a constant and

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_mX_m(t) = \mathbf{0} \text{ for all } t \text{ in } J$$

then

$$c_1 = c_2 = \cdots = c_m = 0.$$

Saying that $(X_1, X_2, \ldots, X_m)$ is linearly dependent over J means that it is not linearly independent.

Note. A list of vector valued functions $(X_1, X_2, \ldots, X_m)$ is linearly dependent over J if and only it there is a list of scalars $(c_1, \ldots, c_m)$ **at least one of which is not zero** such that

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_mX_m(t) = \mathbf{0} \text{ for all } t \text{ in } J$$

Definition. Suppose that $(X_1, X_2, \ldots, X_n)$ is a list of n n-dimensional column vector valued functions. Their Wronski matrix

$$M_W[X_1, X_2, \ldots, X_n]$$

is the $n \times n$ matrix of function whose j-th column is $X_j$for $j = 1, 2, \ldots, n$, and their Wronskian

$$W[X_1, X_2, \ldots, X_n]$$

is the determinant of their Wronski matrix. Note that there are **no derivatives** used in the Wronski matrix or the Wronskian.

Example. If

$$X_1(t) = \begin{pmatrix} e^t \\ \sin t \end{pmatrix} \text{ and } X_2(t) = \begin{pmatrix} t^2 \\ \cos t \end{pmatrix}$$

then

$$M_W[X_1, X_2](t) = \begin{pmatrix} e^t & t^2 \\ \sin t & \cos t \end{pmatrix} \text{ and } W[X_1, X_2](t) = e^t \cos t - t^2 \sin t.$$

Theorem. (First Wronskian Test) If $W[X_1, X_2, \ldots, X_n](t_0) \neq 0$ for some t_0 in J, then $(X_1, X_2, \ldots, X_n)$ is linearly independent over J.

Theorem. (Second Wronskian Test) If $X_j' = AX_j$ for $j = 1, \ldots, n$ and $W[X_1, X_2, \ldots, X_n](t_0) = 0$ for some t_0 in J, then $(X_1, X_2, \ldots, X_n)$ is linearly dependent over J

Definition. Suppose that A is $n \times n$. A **fundamental list** (or set) for $X' = AX$ is a linearly independent list of n n-dimensional column vector valued functions $(X_1, X_2, \ldots, X_n)$ such that $X_j' = AX_j$ for $j = 1, \ldots, n$.

Theorem. (All Solutions to $X' = AX$). Suppose that A is an $n \times n$ matrix of continuous functions defined on an interval J and $(X_1, X_2, \ldots, X_n)$ is a fundamental list for $X' = AX$. It follows that X is a solution to $X' = AX$ if and only if

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n.$$

for some list of scalars $c_1, c_2, \ldots, c_n$. In the text, this last expression is called the **general solution** to $X' = AX$.

Definition. When A is $n \times n$, a **fundamental matrix** for $X' = AX$ is an $n \times n$ matrix of functions whose columns form a fundamental list for $X' = AX$.

Note. Multiplying a matrix on the right by a column vector of the correct dimension gives a linear combination of the columns of the matrix. Thus we have the following alternate way to describe all solutions to $X' = AX$. Suppose that A is an $n \times n$ matrix of continuous functions defined on an interval J and Φ is a fundamental matrix for $X' = AX$. It follows that X is a solution to $X' = AX$ if and only if

$$X = \Phi C$$

for some n-dimensional constant column vector C.

Note. (Initial Value Problems) Suppose that a fundamental list $(X_1, X_2, \ldots, X_n)$ is known for the equation $X' = AX$ on J, E is an n-dimensional constant column vector, and t_0 is a number in J. The solution X to the initial value problem

$$X' = AX \text{ on } J \text{ and } X(t_0) = E$$

is given by

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

where the coefficients $(c_1, c_2, \ldots, c_n)$ are determined by

$$c_1X_1(t_0) + c_2X_2(t_0) + \cdots + c_nX_n(t_0) = E.$$

This is equivalent to

$$\Phi(t_0)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = E$$

where Φ is the fundamental matrix whose columns are $X_1, X_2, \ldots, X_n$. Fundamental matrices are always invertible so

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \Phi^{-1}(t_0)E.$$

In term of Φ, the solution to the initial value problem

$$X' = AX \text{ on } J \text{ and } X(t_0) = E$$

is given by

$$X(t) = \Phi(t)\Phi^{-1}(t_0)E.$$

Example. Consider the system

$$X' = AX \text{ on } (0, \infty)$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ 3/t^2 & 1/t & . \end{pmatrix}.$$

Let

$$X_1(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix} \text{ and } X_2(t) = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$$

Then

$$X_1'(t) = \begin{pmatrix} 3t^2 \\ 6t \end{pmatrix} = A(t)X_1(t) \text{ and } X_2'(t) = \begin{pmatrix} -t^{-2} \\ 2t^{-3} \end{pmatrix} = A(t)X_2(t)$$

$$W[X_1, X_2](t) = \det\begin{pmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{pmatrix} = -4t \text{ so } W[X_1, X_2](1) = -4 \neq 0$$

Thus (X_1, X_2) is a fundamental pair for $X' = AX$ and X is a solution if and only if

$$X(t) = c_1X_1(t) + c_2X_2(t) = c_1\begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix} + c_2\begin{pmatrix} -t^{-2} \\ 2t^{-3} \end{pmatrix}$$

for some pair of numbers $c_1$and c_2.

The fundamental matrix formed by (X_1, X_2) is Φ where

$$\Phi(t) = \begin{pmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{pmatrix}.$$

Note that

$$\Phi^{-1}(t) = \frac{1}{-4t}\begin{pmatrix} -t^{-2} & -t^{-1} \\ -3t^2 & t^3 \end{pmatrix} = \begin{pmatrix} 1/4t^3 & 1/4t^2 \\ 3t/4 & -t^2/4 \end{pmatrix}$$

So

$$\Phi^{-1}(1) = \begin{pmatrix} 1/4 & 1/4 \\ 3/4 & -1/4 \end{pmatrix}.$$

The solution to the initial value problem

$$X' = AX \text{ and } X(1) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is given by

$$
\begin{aligned}
X(t) &= \Phi(t)\Phi^{-1}(1)\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{pmatrix}\begin{pmatrix} 1/4 & 1/4 \\ 3/4 & -1/4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 5/4t - t^3/4 \\ -5/4t^2 - 3t^2/4 \end{pmatrix}.
\end{aligned}
$$

Additional Examples. See the text and the material that is posted online.

Suggested Problems. Do the odd numbered problems for Section 6.2.