Engineering Mathematics

Dr. Philip Walker

Section 6.3
Constant Coefficient Systems - Part I

Note. In this section and the next we will be finding solutions to

$$X' = AX$$

when A is an $n \times n$ constant matrix. We will take the domain interval to be the set of all real numbers.

Theorem. Suppose that A is an $n \times n$ constant matrix, λ_0 is an eigenvalue, and K is a corresponding eigenvector. If

$$X(t) = e^{\lambda_0 t} K$$

then X is a solution to

$$X' = AX,$$

Proof.

$$\begin{aligned} X'(t) &= \lambda_0 e^{\lambda_0 t} K \text{ and } AX(t) = Ae^{\lambda_0 t} K = e^{\lambda_0 t} AK = e^{\lambda_0 t} \lambda_0 K \\ &= \lambda_0 e^{\lambda_0 t} K \end{aligned}$$

Theorem. Suppose that A is an $n \times n$ constant matrix and A has n linearly independent eignevectors $K_1, K_2, \ldots, K_n$ with K_j corresponding to an eigenvalue λ_j for $j = 1, 2, \ldots, n$. (This will always be the case when A has n eigenvalues.) Let

$$X_j(t) = e^{\lambda_j t} K_j \text{ for } j = 1, 2, \ldots, n.$$

It follows that $(X_1, X_2, \ldots, X_n)$ is a fundamental list for

$$X' = AX.$$

Proof. We have just seen that $X_j' = AX_j$ for $j = 1, 2, \ldots, n$. Since $e^0 = 1$, $M_W[X_1, X_2, \ldots, X_n](0)$ is the $n \times n$ matrix whose j-th column is K_j for$1, 2, \ldots, n$. These columns are linearly independent so $W[X_1, X_2, \ldots, X_n](0) \neq 0$. This establishes the linear independence of $(X_1, X_2, \ldots, X_n)$.

Example. Suppose that

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

The eigenvalues of A are 1 and 2. An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and an eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ 2. \end{pmatrix}$. Thus a fundamental pair for $X' = AX$ is (X_1, X_2) where

$$X_1(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } X_2(t) = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and X is a solution to $X' = AX$ if and only if

$$X(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Example. Suppose that

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial $\mathcal{P}$ for A is given by

$$\mathcal{P}(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6.$$

The sum of the coefficients is 0 so the number 1 is a zero of $\mathcal{P}$. Dividing $\mathcal{P}(\lambda)$ by $\lambda - 1$ produces a quadratic quotient which can be factored showing that

$$\mathcal{P}(\lambda) = -(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

Computation shows that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1, $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 2, and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 3.

Thus a fundamental triple for $X' = AX$ is (X_1, X_2, X_3) where

$$X_1(t) = e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; X_2(t) = e^{2t} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \text{ and } X_3(t) = e^{3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

and $X' = AX$ if and only if

$$X(t) = c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Additional Examples. See the text and the material that is posted online..

Suggested Problems. Do the odd numbers for Section 6.3.