Section 6.4

Section 6.4
Constant Coefficient Systems - Part II

Note. In this section we consider two complications that can arise when solving constant coefficient systems. The first complication is that the coefficient matrix can have complex eigenvalues. The second is that the $n \times n$ coefficient matrix may have fewer than n eignevalues.

Complex Eigenvalues

Note. Suppose that the real $n \times n$ matrix A has a non real complex eigenvalue λ_0 and a corresponding eigenvector K_0. Then $\overline{\lambda_0}$ will be an eigenvalue, $\overline{K_0}$ will be a corresponding eigenvector, and the functions whose values at t are

$$e^{\lambda_0 t}K_0 \text{ and } e^{\overline{\lambda_0}t}\overline{K_0}$$

will be independent solutions to

$$X' = AX.$$

This pair can and should be replaced with the real valued pair of functions whose values at t are

$$\mathrm{Re}(e^{\lambda_0 t}K_0) \text{ and } \mathrm{Im}(e^{\lambda_0 t}K_0).$$

These functions will also be linearly independent.

To find these real and imaginary parts suppose that

$$\lambda_0 = \alpha + \beta i \text{ and } K_0 = L + iM$$

where each of α and β is real and each of L and M is an n-dimensional column vector with real entries.

Noting that

$$e^{(\alpha+\beta i)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

we have that

$$\begin{aligned} e^{\lambda_0 t}K_0 &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(L + iM) \\ &= e^{\alpha t}(\cos\beta tL - \sin\beta tM) + ie^{\alpha t}(\cos\beta tM + \sin\beta tL) \end{aligned}$$

Thus

$$\mathrm{Re}(e^{\lambda_0 t}K_0) = e^{\alpha t}(\cos\beta t L - \sin\beta t M)$$
and $$\mathrm{Im}(e^{\lambda_0 t}K_0) = e^{\alpha t}(\cos\beta t M + \sin\beta t L)$$

Example. Consider the equation

$$X' = AX \text{ where } A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}.$$

The characteristic polynomial for A is $\mathcal{P}$ where $\mathcal{P}(\lambda) = (-1-\lambda)(-1-\lambda)+4 = \lambda^2+2\lambda+5$. The quadratic formula shows that the zeros of $\mathcal{P}$, hence the eigenvalues of A, are $-1+2i$ and $-1-2i$.

$$[A-(-1+2i)I]K = \mathbf{0} \text{ is equivalent to} \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The augmented matrix for this equation is

$$\begin{pmatrix} -2i & -4 & 0 \\ 1 & -2i & 0 \end{pmatrix} \text{ whose RREF is } \begin{pmatrix} 1 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ is in the eigenspace if and only if $k_1 - 2ik_2 = 0$ or, setting $k_2 = a$, $K = a\begin{pmatrix} 2i \\ 1 \end{pmatrix}$ for some number a. Taking $a = 1$, we see that an eigenvector corresponding to the eigenvalue $-1+2i$ is $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$. A complex valued solution to $X' = AX$ is the function whose value at t is

$$e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

$$e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t}e^{2it}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$
$$= e^{-t}(\cos 2t + i\sin 2t)\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]$$

$$= e^{-t}\left[\cos 2t\begin{pmatrix}0\\1\end{pmatrix} - \sin 2t\begin{pmatrix}2\\0\end{pmatrix}\right] + ie^{-t}\left[\cos 2t\begin{pmatrix}2\\0\end{pmatrix} + \sin 2t\begin{pmatrix}0\\1\end{pmatrix}\right]$$

A fundamental pair of real valued functions for $X' = AX$ is (X_1, X_2) where

$$X_1(t) = e^{-t}\left[\cos 2t\begin{pmatrix}0\\1\end{pmatrix} - \sin 2t\begin{pmatrix}2\\0\end{pmatrix}\right]$$

and

$$X_2(t) = e^{-t}\left[\cos 2t\begin{pmatrix}2\\0\end{pmatrix} + \sin 2t\begin{pmatrix}0\\1\end{pmatrix}\right].$$

X is a solution to $X' = AX$ if and only if

$$X = c_1X_1 + c_2X_2$$

for some pair of scalars c_1 and c_2.

Example. Consider the equation

$$X' = Ax \text{ where } A = \begin{pmatrix}1 & -4 & -1\\3 & 2 & 3\\1 & 1 & 3\end{pmatrix}.$$

The characteristic polynomial $\mathcal{P}$ for A is given by

$$\mathcal{P}(\lambda) = -\lambda^3 + 6\lambda^2 - 21\lambda + 26.$$

By inspection, 2 is a zero of $\mathcal{P}$ and dividing $\lambda - 2$ into $\mathcal{P}(\lambda)$ produces a quotient of $-\lambda^2 + 4\lambda - 13$ Thus

$$\mathcal{P}(\lambda) = -(\lambda - 2)(\lambda^2 - 4\lambda + 13)$$

Focusing on $(\lambda^2 - 4\lambda + 13)$, the quadratic formula shows that $2 + 3i$ and $2 - 3i$ are also zeros of $\mathcal{P}$. The eigenvalues of A are 2, $2 + 3i$, and $2 - 3i$.

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ so one solution to $X' = AX$ is X_1 where

$$X_1(t) = e^{2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

An eigenvector corresponding to the eigenvalue $2 + 3i$ is $\begin{pmatrix} -5+3i \\ 3+3i \\ 2 \end{pmatrix}$ so a complex valued solution to $X' = AX$ is U where

$$\begin{aligned} U(t) &= e^{(2+3i)t}\begin{pmatrix} -5+3i \\ 3+3i \\ 2 \end{pmatrix} \\ &= e^{2t}(\cos 3t + i\sin 3t)\left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}\right] \end{aligned}$$

$$\operatorname{Re} U(t) = e^{2t}\left[\cos 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}\right]$$

and

$$\operatorname{Im} U(t) = e^{t}\left[\cos 3t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}\right].$$

A fundamental list for $X' = AX$ is (X_1, X_2, X_3) where

$$X_1(t) = e^{2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, X_2(t) = e^{2t}\left[\cos 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}\right]$$

and

$$X_3(t) = e^t\left[\cos 3t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}\right]$$

The $n \times n$ Coefficient Matrix Has Fewer Than n Eigenvalues

The Coefficient Matrix is a 2×2 Diagonal Matrix.

Note. If

$$A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

then there is only one eigenvalue, namely λ_0, and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are independent corresponding eigenvectors. A fundamental pair for $X' = AX$ is (X_1, X_2) where

$$X_1(t) = e^{\lambda_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } X_2(t) = e^{\lambda_0 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$X' = AX \text{ if and only if } X(t) = c_1 e^{\lambda_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_0 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Coefficient Matrix is 2×2, Has Only One Eigenvalue, and is Not a Diagonal Matrix.

Note. Suppose that A is 2×2, is not a diagonal matrix, and has only one eigenvalue λ_0. In this case, the eigenspace will be one dimensional. Let K be an eigenvector corresponding to λ_0, and let W be a two dimensional column vector satisfying

$$(A - \lambda_0 I)W = K.$$

Each such vector W is called a generalized eigenvector. There will be infinitely many of them, but you need only one. In this case, a fundamental pair for $X' = AX$ is (X_1, X_2) where

$$X_1(t) = e^{\lambda_0 t}K \text{ and } X_2(t) = e^{\lambda_0 t}(tK + W).$$

Example. Suppose that $A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$. Then A has only one eigenvalue, namely 2.

Solving

$$(A - 2I)K = \mathbf{0} \text{ or } \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see that the eigenspace is one dimensional, and that an eigenvector corresponding to the eigenvalue 2 is K where

$$K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The equation $(A - 2I)W = K$ is

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ where } W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The augmented matrix for this system is

$$\begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so P is a solution to $(A - 2I)W = K$ if and only if

$$W = \begin{pmatrix} \frac{1}{2}a - \frac{1}{2} \\ a \end{pmatrix}$$

for some number a. One such vector W is given by taking $a = 1$ so that

$$W = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A fundamental pair for $X' = AX$ is (X_1, X_2) where

$$X_1(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ and } X_2(t) = e^{2t}\left[t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

X is a solution to $X' = AX$ if and only if

$$X = c_1X_1 + c_2X_2.$$

The Coefficient Matrix A is 3×3, **Has Only Two Eigenvalue**, **the Characteristic Polynomial** $\mathcal{P}$ **is Given by** $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ **where** $\lambda_1 \neq \lambda_2$ **and** $\text{rank}(A - \lambda_2 I) = 1$.

Note. If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ where $\lambda_1$and λ_2 are distinct real numbers so that A has an eigenvalue λ_1 of algebraic multiplicity 1 and an eigenvalue λ_2 of algebraic multiplicity 2, and

$$\text{rank}(A - \lambda_2 I) = 1,$$

(This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly two all zero rows.) let K_1 be an eigenvector corresponding to λ_1 and let K_2 and K_3 be independent eigenvectors corresponding to λ_2. Then let

$$x_1(t) = e^{\lambda_1 t}K_1,\ x_2(t) = e^{\lambda_2 t}K_2,\ \text{and}\ x_3(t) = e^{\lambda_2 t}K_3.$$

Example. Consider the system

$$X' = AX$$

where

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

The characteristic polynomial $\mathcal{P}$ is given by

$$\mathcal{P}(\lambda) = -(\lambda - 4)(\lambda + 2)^2.$$

$$(A - 4I) = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvectors are of the form $\begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ a \end{pmatrix}$.

One eigenvector corresponding to the eigenvalue 4 is K_1 where

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$$A - (-2)I = A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

K is in the eigenspace corresponding to -2if and only if

$$K = \begin{pmatrix} b-a \\ a \\ b \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

for some pair of numbers a and b.

An independent pair (K_2, K_3) of eigenvectors corresponding to -2 is obtained by first letting $a = 1$ and $b = 0$ then letting $a = 0$ and $b = 1$so that

$$K_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } K_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

A fundamental triple for $X' = AX$ is (X_1, X_2, X_3) where

$$X_1(t) = e^{4t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, X_2(t) = e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \text{ and } X_3(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$X' = AX$ if and only if

$$X = c_1X_1 + c_2X_2 + c_3X_3.$$

The Coefficient Matrix A is 3×3, **Has Only Two Eigenvalue**, **the Characteristic Polynomial** $\mathcal{P}$ **is Given by** $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ **where** $\lambda_1 \neq \lambda_2$ **and** $\operatorname{rank}(A - \lambda_2 I) = 2$.

NOTE. If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ where $\lambda_1$and λ_2 are distinct real numbers so that A has an eigenvalue λ_1 of algebraic multiplicity 1 and an eigenvalue λ_2 of algebraic multiplicity 2, and

$$\operatorname{rank}(A - \lambda_2 I) = 2,$$

(This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly one all zero rows.) let K_1 be an eigenvector corresponding to λ_1, K_2 be an eigenvector corresponding to λ_2 and let W be a three-dimensional column vector satisfying

$$(A - \lambda_2 I)W = K_2.$$

(W is called a generalized eigenvector.) Then let

$$x_1(t) = e^{\lambda_1 t}K_1,\ x_2(t) = e^{\lambda_2 t}K_2, \text{ and } x_3(t) = e^{\lambda_2 t}[tK_2 + W].$$

Example. Consider the system $X' = AX$ where$A = \begin{pmatrix} -6 & -7 & -13 \\ 5 & 6 & 9 \\ 2 & 2 & 5 \end{pmatrix}$.

The characteristic polynomial $\mathcal{P}$ is given by

$$\mathcal{P}(\lambda) = -(\lambda - 3)(\lambda - 1)^2.$$

$$A - 3I = \begin{pmatrix} -9 & -7 & -13 \\ 5 & 3 & 9 \\ 2 & 2 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} -9 & -7 & -13 & 0 \\ 5 & 3 & 9 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

An eigenvector K_1 corresponding to the eigenvalue 3 is given by

$$K_1 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

$$A - (1)I = \begin{pmatrix} -7 & -7 & -13 \\ 5 & 5 & 9 \\ 2 & 2 & 4 \end{pmatrix}$$

and

$$\begin{pmatrix} -7 & -7 & -13 & 0 \\ 5 & 5 & 9 & 0 \\ 2 & 2 & 4 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

An eigenvector K_2 corresponding to the eigenvalue 1 is given by

$$K_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Column vectors W such that

$$(A - (1)I)W = K_2 \text{ or } \begin{pmatrix} -7 & -7 & -13 \\ 5 & 5 & 9 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

are given by

$$W = \begin{pmatrix} 2-a \\ a \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

We need only one solution W so we will let $a = 0$ and use

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

A fundamental triple for $X' = AX$ is (X_1, X_2, X_3) where

$$X_1(t) = e^{3t}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \; X_2(t) = e^{1}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$X_3(t) = e^{t}\left[t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right].$$

Additional Examples. See the text and the material that is posted online.

Suggested Problems. Do the odd numbers for Section 6.4.